DE SVMMATIONE INNVMERABILIVM PROGRESSIONVM.

Auct. L. *Eulero.*

I.

QVae in praecedente diſſertatione de progreſſionibus transcendentibus earumque terminis generalibus tradidi, multo latius patent, quam videri poſſent; et inter alia quam plurima, ad quae accommodari poſſunt, eximius earum poteſt eſſe vſus in inueniendis ſummis innumerabilium progreſſionum. Quemadmodum enim in ſuperiore diſſertatione innumerae progreſſiones ad terminos generales ſunt reuocatae, quae communem algebram transcendunt; ita hic eandem methodum accommodabo ad terminos ſummatorios inueniendos progreſſionum, ad quas indefinite ſummandas communis algebra non ſufficit.

§. 2. Progreſſio quaepiam ſummari dicitur indefinite, ſi detur formula numerum indefinitum n continens, quae exponat ſummam tot terminorum illius progreſſionis, quot n comprehendit vnitates, ita vt ſi ponatur v. gr. $n=10$ ea formula exhibeat ſummam decem terminorum a primo numeratorum. Formula haec vocatur terminus ſummatorius illius progreſſionis; atque eſt ſimul terminus generalis progreſſionis, cuius terminus quicunque aequatur ſummae tot terminorum illius progreſſionis, quot eius exponens in ſe continet vnitates.

§. 3. Cum progreſſiones quaeque exponantur terminis generalibus, quaeſtio de ſummandis progreſſionibus eſt haec, vt ex termino generali terminus ſummatorius inueniatur. Etquidem iam eo eſt peruentum, vt, quoties terminus generalis eſt functio rationalis ipſius indicis n exponentes ſunt numeri integri affirmatiui, ſemper terminus ſummatorius inueniri queat. Quando autem exponentes ipſius n ſunt negatiui, niſi excipiantur pauci caſus, nemo adhuc terminos ſummatorios dedit. Ratio huius difficultatis eſt, quod tum termini ſummatorii plerumque algebraice exprimi nequeant, ſed tales requirant formas, quae quadraturas in ſe contineant.

§. 4. Aſſumatur haec forma $\int\frac{1-x^n}{1-x}dx$, tanquam terminus generalis cuiusdam progreſſionis; quae ſcilicet integrata, ita vt fiat $=0$ ſi $x=0$ poſitoque $x=1$ daret terminum ordine n. Progreſſio quae hoc modo ex ea formatur erit haec 1, $1+\frac{1}{2}$, $1+\frac{1}{2}+\frac{1}{3}$, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, etc. cuius ergo terminus generalis eſt formula aſſumta $\int\frac{1-x^n}{1-x}dx$

Series

Series vero haec inuenta summatoria est progressionis harmonicae $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ etc. cuius terminus generalis est $\frac{1}{n}$. Quamobrem huius progressionis terminus summatorius erit $\int \frac{1-x^n}{1-x}\,dx$, qui illius est terminus generalis.

§. 5. Cum terminus generalis progressionis 1, $1+\frac{1}{2}$, $1+\frac{1}{2}+\frac{1}{3}$, etc. sit $\int \frac{1-x^n}{1-x}\,dx$. Poterit ex hoc ea progressio interpolari, seu quilibet terminus medius inueniri: vt si requiratur terminus, cuius index est $\frac{1}{2}$, oportebit integrari $\frac{1-\sqrt{x}}{1-x}\,dx$ vel $\frac{dx}{1+\sqrt{x}}$, cuius integrale est $2\sqrt{x}-2\,l(1+\sqrt{x})$ quod cum fiat $=o$ si $x=o$, ponatur $x=1$, erit terminus ordine $\frac{1}{2}=2-2\,l\,2$. Deinde, quia generaliter terminus ordine $n+1$, terminum ordine n superat fractione $\frac{1}{n+1}$, erit terminus ordine $1\frac{1}{2}=2\frac{2}{3}-2\sqrt{2}$, et terminus ordine $2\frac{1}{2}$ hic $2+\frac{2}{3}+\frac{2}{5}-2\,l\,2$, etc. Series igitur interpolata erit

$$\begin{array}{ccccc} \frac{1}{2} & 1 & 1\frac{1}{2} & 2 & 2\frac{1}{2} \\ 2-2\,l\,2, & 1, & 2+\frac{2}{3}-2\,l\,2, & 1+\frac{1}{2}, & 2+\frac{2}{3}+\frac{2}{5}-2\,l\,2,\ \text{etc.} \end{array}$$

§. 6. Ad hunc modum rem generalius complexus sum, et assumsi formulam $\int \frac{1-P^n}{1-P}\,dx$, vbi P denotat functionem quamcunque ipsius x. Integrale hoc vt semper ita debet accipi vt posito $x=o$, id totum fiat $=o$. Deinde hoc facto non

non vt ante pono $x=1$, sed vt latius pateat pono $x=k$. Forma hoc modo resultans erit terminus ordine n progressionis cuiusdam, cuius terminus generalis est forma assumta $\int\frac{1-P^n}{1-P}dx$. Progressio vero ipsa haec erit k, $k+\int Pdx$, $k+\int Pdx+\int P^2dx$, etc. vbi in integralibus $\int Pdx$, $\int P^2dx$, etc. loco x iam positum esse k pono.

§. 7. Progressio inuenta, si quiuis terminus a sequente subtrahatur, praebebit hanc k, $\int Pdx$, $\int P^2dx$, $\int P^3dx$ etc. Huiusque terminus summatorius aequalis est termino generali praecedentis progressionis, cuius terminus generalis est $\int P^{n-1}dx$, haec formula $\int\frac{1-P^n}{1-P}dx$. Sit $P=x^\alpha : a^\alpha$ erit progressionis huius k, $\frac{k^{\alpha+1}}{(\alpha+1)a^\alpha}$, $\frac{k^{2\alpha+1}}{(3\alpha+1)a^{2\alpha}}$ etc. terminus generalis $\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}}$ atque terminus summatorius hic $\int\frac{a^{n\alpha}-x^{n\alpha}}{(a^\alpha-x^\alpha)a^{n\alpha-\alpha}}dx$.

§. 8. Inuentus ergo est terminus summatorius pro omnibus progressionibus quorum termini sunt fractiones, harumque numeratores progressionem geometricam, denominatores vero arithmeticam constituunt. Vt vero facilius ad omnes casus accommodari possit, sumatur haec progressio,

ſio, $\frac{b}{c}$, $\frac{b^{i+1}}{c+e}$, $\frac{b^{2i+1}}{c+2e}$, $\frac{b^{3i+1}}{c+3e}$, cuius terminus generalis eſt $\frac{b^{(n-1)i+1}}{c+(n-)e}$, comparetur hic cum illo $\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}}$ vel $\frac{c\,k^{(n-1)\alpha+1}}{(c+(n-1)\alpha c)\,a^{(n-1)\alpha}}$ erit $\alpha = \frac{e}{c}$ et $\frac{c\,k^{\frac{(n-1)e}{c}+1}}{a^{\frac{(n-1)e}{c}}} = b^{(n-1)i+1}$ atque $a =$ $\left(\frac{ck^{\frac{(n-1)e}{c}+1}}{b^{(n-1)i+1}}\right)^{\frac{c}{(n-1)e}} = \left(\frac{ck}{b}\right)^{\frac{c}{(n-1)e}} \frac{k}{b^{ci:e}}$. Hic ne a pendeat ab n, debet enim a eſſe conſtans quantitas, oportet vt $\frac{ck}{b}$ ſit $=1$, erit ergo $k = \frac{b}{c}$, atque $a = \frac{b^{\frac{e-ci}{e}}}{c}$. Quocirca terminus ſummatorius eſt $\int \frac{b^{\frac{ne-nci}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)(e-ci)}{c}}\left(b^{\frac{e-ci}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}}\right)} dx$. Quae ita debet integrari vt fiat $=0$ ſi $x=0$, tum vero ponere oportet $x = \frac{b}{c}$.

§. 9. Cognita ſumma progreſſionis indefinita habebitur ſumma progreſſionis in infinitum, ſi ponatur $n = \infty$. Terminus quidem ſummatorius inuentus non magis ad hunc caſum quam ad aliumquemque accommodatus videtur. Eſt mihi vero alia methodus ſummas ſerierum infinitarum inueſtigandi, quae latiſſime patet. Sit ſeries $\frac{b}{c} + \frac{b^{i+1}}{c+1} +$

b^{2i+1}

$\frac{b^{2i+1}}{c+2e}$ etc. Ponatur numerus terminorum n, et summa eorum A. Augeatur numerus n vnitate, augebitur summa A termino ordine $n+1$, qui est $\frac{b^{ni+1}}{c+ne}$. Si nunc n et A tanquam quantitates fluentes considerentur, quia n est quasi infinities maior quam 1, erunt earum differentialia dn et dA inter se vt augmenta 1 et $\frac{b^{ni+1}}{c+ne}$. Vnde prodit aequatio $dA=\frac{b^{ni+1}dn}{c+ne}$. Quae integrata dabit aequationem inter summam A et numerum terminorum n.

§. 10. Ponatur $l(c+ne)=z$, erit $\frac{edn}{c+ne}=dz$ atque $e+ne=g^z$ denotante g numerum, cuius logarithmus est 1. Est ergo $n=\frac{g^z-c}{c}$ et b^{ni+1} $=b^{\frac{g^z i-ci}{e}+1}=b^{\frac{e-ci}{e}}b^{\frac{g^z i}{e}}$; consequenter $dA=\frac{b^{\frac{e-ci}{e}}}{e}b^{\frac{g^z i}{e}}dz$. Haec quidem aequatio ita generaliter instituta integrationem nisi per series non admittit. Si vero ponatur $i=0$, vt prodeat seriei $\frac{b}{c}+\frac{b}{c+e}+\frac{b}{c+2e}+$ etc., habebitur aequatio dA $=\frac{b}{e}dz$ et $A=\frac{b}{e}z+lC)=\frac{b}{e}lC(c+ne)$. Constans quidem C non determinatur, sed tamen aequa-

aequatio ad definiendam differentiam inter duas fummas inferuit; vt fit alius numerus terminorum m, et fumma B erit, $B = \frac{b}{e} l C(c+me)$. Ergo $A - B = \frac{b}{e} l \frac{c+ne}{c+me} = \frac{b}{e} l \frac{m}{n}$, quia m et n funt infinita.

§. 11. Maneat $i=o$, et progreffio erit haec $\frac{b}{c}, \frac{b}{c+e}, \frac{b}{c+2e}, \frac{b}{c+3e}$ etc. cuius terminus generalis eft $\frac{b}{c+(n-1)e}$. Terminus autem fummatorius eft $\int \frac{b^{\frac{ne}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)e}{c}}(b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})} dx$. Sumatur alia progreffio $\frac{b}{c}, \frac{b}{c+f}, \frac{b}{c+2f}, \frac{b}{c+3f}$ etc. cuius terminus generalis eft $\frac{b}{c+(n-1)f}$ et fummatorius $\int \frac{b^{\frac{nf}{c}} - c^{\frac{nf}{c}} x^{\frac{nf}{c}}}{b^{\frac{(n-1)f}{c}}(b^{\frac{f}{c}} - c^{\frac{f}{c}} x^{\frac{f}{c}})} dx$; in quo integrato itidem ponere oportet $x = k = \frac{b}{c}$. Addantur hae duae progreffiones, fcilicet terminus primus primo, fecundus fecundo, et ita porro prodibit haec progreffio $\frac{2b}{c}, \frac{2bc+b(e+f)}{(c+e)(c+f)}, \frac{2bc+2b(e+f)}{(c+2e)(c+2f)}$ etc., cuius terminus generalis eft $\frac{2bc+(n-1)b(e+f)}{(c+(n-1)e)(c+(n-1)f)}$. Terminus vero fummatorius erit $= \int dx \left(\frac{b^{\frac{ne}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)e}{c}}(b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})} + \frac{b^{\frac{nf}{c}} - c^{\frac{nf}{c}} x^{\frac{nf}{c}}}{b^{\frac{(n-1)e}{c}}(b^{\frac{f}{c}} - c^{\frac{f}{c}} x^{\frac{f}{c}})} \right)$

§. 12. Simili modo, fed vniuerfalius, pro termino generali in cuius denominatore n duas tenet dimenfiones, inuenitur terminus fummatorius, fi illius progreffionis p cuplum ad huius q cuplum addatur. Obtinebitur hoc modo progref-

 fio,

ſio, cuius terminus generalis eſt $\frac{pb}{c+(n-1)e}+\frac{qb}{c+(n-1)f}$ $=\frac{(p+q)bc+(n-1)b(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$. Terminus autem ſummatorius huic termino generali reſpondens erit

$$\int\frac{p\,dx}{b^{\frac{(n-1)e}{c}}}\left(\frac{b^{\frac{ne}{c}}-c^{\frac{ne}{c}}x^{\frac{ne}{c}}}{b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}}}\right)+\int\frac{q\,dx}{b^{\frac{(n-1)f}{c}}}\left(\frac{b^{\frac{nf}{c}}-c^{\frac{nf}{c}}x^{\frac{nf}{c}}}{b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}}}\right)$$

$$=\int dx\left(\frac{pb^{\frac{n(e+f)}{c}}-pb^{\frac{n(e+f)-f}{c}}c^{\frac{f}{c}}x^{\frac{f}{c}}-pb^{\frac{nf}{c}}c^{\frac{ne}{c}}x^{\frac{ne}{c}}+pb^{\frac{(n-1)f}{c}}c^{\frac{ne+f}{c}}x^{\frac{ne+f}{c}}+qb^{\frac{n(e+f)}{c}}-qb^{\frac{n(e+f)-e}{c}}c^{\frac{e}{c}}x^{\frac{e}{c}}-qb^{\frac{ne}{c}}c^{\frac{nf}{c}}x^{\frac{nf}{c}}+qb^{\frac{(n-1)e}{c}}c^{\frac{nf+e}{c}}x^{\frac{nf+e}{c}}}{b^{\frac{(n-1)(e+f)}{c}}\,(b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}})\,(b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}})}\right).$$

Ponatur $b=1$ hoc enim modo vniuerſalitati nihil decedit, eritque terminus generalis $\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$. Sit $cx=y$, erit $dx=\frac{dy}{c}$. Atque terminus ſummatorius habetur

$$=\int\frac{dy}{c}\left(\frac{p+q-py^{\frac{f}{c}}-qy^{\frac{e}{c}}-py^{\frac{ne}{c}}-qy^{\frac{nf}{c}}+qy^{\frac{ne+f}{c}}+qy^{\frac{nf+e}{c}}}{(1-y^{\frac{e}{c}})(1-y^{\frac{f}{c}})}\right)$$

in qua formula integrata, ita vt poſito $y=0$ ea quoque fiat $=0$, oportet ponere $y=1$.

§. 13. Aſſumatur iam terminus generalis hic $\frac{\alpha+\beta n}{\gamma+\delta n+\varepsilon nn}$. Qui comparatus cum $\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$ dabit $c=\sqrt{(\gamma+\delta+\varepsilon)}$, $e=\frac{\delta+2\varepsilon+\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)}}$, $f=\frac{\delta+2\varepsilon-\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)}}$ $p=\frac{\alpha\delta-\beta\delta+2\alpha\varepsilon-2\beta\gamma+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)}\,(\delta\delta-4\gamma\varepsilon)}$ atque $q=\frac{\beta\delta-\alpha\delta+2\beta\gamma-2\alpha\varepsilon+(\alpha+\beta)\sqrt{\delta\delta-4\gamma\varepsilon}}{2\sqrt{(\gamma+\delta+\varepsilon)}(\delta\delta-4\gamma\varepsilon)}$. His in termino ſummatorio ſubſtitutis, prodibit terminus ſummatorius

huius

huius progressionis $\frac{\alpha+\beta}{\gamma+\delta+\varepsilon}$, $\frac{\alpha+2\beta}{\gamma+2\delta+4\varepsilon}$, $\frac{\alpha+3\beta}{\gamma+3\delta+9\varepsilon}$, etc. cuius terminus generalis est $\frac{\alpha+\beta n}{\gamma+\delta n+\varepsilon nn}$.

§. 13. Eodem modo si in termino generali n plures duabus dimensiones habuerit, eruetur terminus summatorius, combinandis tot progressionibus simplicibus, quot dimensiones n habere debet, quemadmodum idem in casu duarum dimensionum factum est. Attamen hac ratione non ad quasuis, quae in huiusmodi terminis generalibus contineri videntur, series perueniri potest. Nam quoties denominator $\gamma+\delta n+\varepsilon n^2+\zeta n^3+\eta n^4$ etc. duos pluresue habet factores simplices aequales, tum progressio in tot simplices progressiones resolui nequit, neque igitur eius terminus summatorius inueniri.

§. 14. Hanc ob rem aliam tradam methodum, quae hos casus non excludat. Sit progressio quaedam simplex $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{1}{a+2b}$ etc. cuius terminus generalis est $\frac{1}{a+(n-1)b}$. Huius terminus summatorius erit $\int \frac{1-a^{\frac{nb}{a}}x^{\frac{nb}{a}}}{1-a^{\frac{b}{a}}x^{\frac{b}{a}}}dx$, vel ponatur $ax = y$, erit is $\int \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\cdot\frac{dy}{a}$, in quo integrato poni oportet $y=1$. Multiplicetur hic in $y^\alpha dy$ et summa huius facti $\int y^\alpha dy \int \frac{dy}{a}\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)$ erit secundum

 modum

modum deſcriptum tractata terminus ſummatorius huius progreſſionis $\frac{a}{a\beta a}$, $\frac{a}{(a+b)(\beta a+b)}$, $\frac{a}{(a+2b)(\beta a+2b)}$ etc. poſito breuitatis ergo β loco $\alpha+2$. Huius progreſſionis terminus generalis eſt $\frac{a}{(a+(n-1)b)(\beta a+(n-1)b)}$ vel

$$\frac{a}{b^2n^2+(ab+\beta ab-2bb)n+(a-b)(\beta a-b)}.$$

§. 15. Aſſumamus progreſſionem generalem huius generis, quae facilius ad caſus quosuis adaptatur, ſit eius terminus generalis $\frac{1}{a+(n-1)b+\frac{(n-1)(n-2)}{2}c}$. Hic cum illo termino generali comparatus dabit $a=\frac{(2b-c)^2-4ac+(2b-c)\sqrt{(2b-c)^2-8ac)}}{4c}$ $b=\frac{2b-c+\sqrt{(2b-c)^2-8ac)}}{4}$, $\beta=\frac{2b-c-\sqrt{(2b-c)^2-8ac}}{2b-c+\sqrt{(2b-c)^2-8ac}}$. Hi valores ſi ſubſtituantur loco a, b, et α, (eſt vero $\alpha=\beta-2$) in $\int y^\alpha dy\int\frac{dy}{a}\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)$ prodibit terminus ſummatorius progreſſionis propoſitae, $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{1}{a+2b+c}$, $\frac{1}{a+3b+3c}$ etc.

§. 16 Hoc modo vlterius progredi licet; multiplicetur $\int\frac{dy}{a}\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)$ in $y^{\alpha-2}dy$, et facti integrale $\int y^{\alpha-2}dy\int\frac{dy}{a}\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)$ denuo in $y^{\beta-\alpha-1}$ huiusque producti integrale $\int y^{\beta-\alpha-1}dy\int y^{\alpha-2}dy\int\frac{dy}{a}$

$(1-y$

$\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)$, erit terminus summatorius progressionis huius $\frac{a^2}{a.\alpha a.\beta a}$, $\frac{a^2}{(a+b)(\alpha a+b)(\beta a+b)}$, $\frac{a^2}{(a+2b)(\alpha a+2b)(\beta a+2b)}$ etc. cuius terminus generalis est $a^2:(a+(n-1)b)(\alpha a+(n-1)b)(\beta a+(n-1)b)$ similiter $\int y^{\gamma-\beta-1}dy\int y^{\beta-\alpha-1}dy\int y^{\alpha-2}dy\int\frac{dy}{a}\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)$ est terminus summatorius progressionis, cuius terminus generalis est $\frac{a^3}{(a+(n-1)b)(\alpha a+(n-1)b)(\beta a+(n-1)b)(\gamma a+(n-1)b)}$. Hoc igitur modo ad omnes progressiones peruenitur, quarum termini sunt fractiones, numeratoribus existentibus numeris constantibus, denominatoribus autem constituentibus quamcunque progressionem algebraicam.

§. 17. Si summae huiusmodi progressionum in infinitum continuatarum desiderentur, oportet ponere $n =$ infinito. Hoc posito postremum cuiusque termini summatorii membrum scilicet $\int\frac{dy}{a}\left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\right)$ transmutabitur in hoc $\int\frac{dy}{a(1-y^{\frac{b}{a}})}$. Quia enim y semper est minus quam 1, praeter casum ultimum, quo fit $y=1$, euanescet $y^{\frac{nb}{a}}$ prae 1: atque ideo $1-y^{\frac{nb}{a}}$ abibit in 1. Propterea huius seriei $\frac{a}{a.\alpha a}+\frac{a}{(a+b)(\alpha a+b)}+\frac{a}{(a+2b)(\alpha a+2b)}+$ etc. in infinitum summa erit $\int y^{\alpha-2}dy\int\frac{dy}{a(1-y^{\frac{b}{a}})}$, et hu-

huius $\frac{a^2}{a.\,\alpha a.\,\beta a} + \frac{a^2}{(a+b)(\alpha a+c)(\beta a+c)} + \frac{a^2}{(a+2b)(\alpha a+2b)(\beta a+2b)}$ + etc., ſumma erit $\int y^{\beta-\alpha-1} y \int y^{\alpha-2} dy \int \frac{dy}{a(1-y^{\frac{b}{a}})}$ et ita de reliquis omnibus.

§. 18. Sit $b=a$, vt fiat $\frac{b}{a}=1$, erit $\int \frac{dy}{a(1-y)} = A - \frac{1}{a} l(1-y)$. Quia poſito $y=o$ totum integrale fieri debet $=0$, erit $A=0$, adeoque $\int \frac{dy}{a(1-y)} = -\frac{1}{a} l(1-y)$. Multiplicetur hoc in $y^{\alpha-2} dy$ habebitur $-\frac{y^{\alpha-2}dy}{a} l(1-y)$. Huius integrale vt inueniatur ponatur $1-y=z$, erit $y=1-z$, habebitur igitur integrandum $\frac{(1-z)^{\alpha-2}dz}{a} lz = (1 - \frac{(\alpha-2)}{1} z + \frac{(\alpha-2)(\alpha-3)}{1.\,2} z^2 - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.\,2.\,3} z^3 + \text{etc.}) \frac{dz}{a} lz$. Quia vero $z^\eta dz lz = C - \frac{z^{\eta+1}}{(\eta+1)^2} + \frac{z^{\eta+1} lz}{\eta+1}$, erit illius integrale haec ſeries $\frac{1}{a}(C - z + z lz + \frac{(\alpha-2)}{1.4} z^2 - \frac{(\alpha-2)}{1.2} z^2 lz - \frac{(\alpha-2)(\alpha-3)}{1.\,2.\,9} z^3 + \frac{(\alpha-2)(\alpha-3)}{1\,2.\,3.} z^3 lz + \text{etc.})$ Hoc integrale ſi fiat $y=0$ ſeu $z=1$ debet fieri $=0$, hanc ob rem erit $C = 1 - \frac{(\alpha-2)}{1.\,4.} + \frac{(\alpha-2)(\alpha-3)}{1.\,2.\,9} - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.\,2.\,3.\,16}$ etc.

§. 19. Perſpicuum eſt ex hoc integrali, quoties α ſit numerus integer vnitate maior, tum ſemper integralis eius terminorum numerum fore finitum, atque ideo ſummam progreſſionis definiri. Attamen etiamſi terminorum numerus ſit infi-

infinitus, ſumma propoſitae ſeriei dabitur per aliam ſeriem infinitam quae vero plerumque magis conuergit quam propoſita, atque ideo perquam eſt vtilis ad ſummam determinandam.

§. 20. Sit ſumma progreſſionis in infinitum continuatae $\int \frac{-y^{\alpha-2}}{a} dy\, l(1-y)$, quia hic eſt poſitum $b = a$, erit progreſſio ipſa $\frac{1}{\alpha a} + \frac{1}{2(\alpha+1)a} + \frac{1}{3(\alpha+3)a} + \frac{1}{4(\alpha+4)a}$ etc. Huius ſumma habetur ſi in illo integrali ponitur $y = 1$, ſed facto $y = 1 - z$ eſt integrale illud $\frac{1}{\alpha}\left(1 - \frac{(\alpha-2)}{1.4} + \frac{(\alpha-2)(\alpha-3)}{1.2.9} - \text{etc.} - z + \frac{(\alpha-2)}{1.4} z^2 - \frac{(\alpha-2)(\alpha-3)}{1.2.9} z^3 + \text{etc.} + z\,lz - \frac{(\alpha-2)}{1.2} z^2\,lz + \frac{(\alpha-2)(\alpha-3)}{1.2.3.} z^3\,lz - \text{etc.}\right)$. Si iam fiat $y = 1$ vel $z = 0$ erit ſumma ſeriei $\frac{1}{\alpha a} + \frac{1}{2(\alpha+1)a} + \frac{1}{3(\alpha+2)a}$ etc. aequalis ſummae huius ſeriei $\frac{1}{a} - \frac{(\alpha-2)}{1.4.a} + \frac{(\alpha-2)(\alpha-3)}{1.2.9.a}$ etc. vel ſumma huius $\frac{1}{\alpha} + \frac{1}{2(\alpha+1)} + \frac{1}{3(\alpha+2)}$ etc. aequalis ſummae huius $1 - \frac{(\alpha-2)}{1.4} + \frac{(\alpha-2)(\alpha-3)}{1.2.9}$ — etc.

§. 21. Praeterea alium habeo modum ſeries valde conuergentes inueniendi, quarum ſumma aequalis ſit ſeriei propoſitae. $\int -y^{\alpha-2} dy\, l(1-y)$ aequatur ita integratum vt fiat $= 0$ ſi $y = 0$ huic ſeriei $1 - \frac{(\alpha-2)}{1.4} + \frac{(\alpha-2)(\alpha-3)}{1.2.9} - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3\ 16} + \text{etc.} - z + \frac{(\alpha-2)}{1.4} z^2 - \frac{(\alpha-2)(\alpha-3)}{1.2.9} z^3 + \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.16} z^4 - \text{etc.} + z\,lz - \frac{(\alpha-2)}{1.2} z^2\,lz + \frac{(\alpha-2)(\alpha-3)}{1.2.3} z^3\,lz - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.4} z^4\,lz + \text{etc.}$ exiſtente $z = 1 - y$; ſed cum ſit $-l(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4}$ etc. erit $\int -y^{\alpha-2} dy\, l(1-y)$ $=$

$= \frac{y^{\alpha}}{\alpha} + \frac{y^{\alpha+1}}{2(\alpha+1)} + \frac{y^{\alpha+2}}{3(\alpha+2)}$ etc. Haec series si a est numerus affirmatiuus est aequalis illi quicquid sit y, et ita multis modis series eiusdem summae reperiuntur, quarum altera alterius ope facilius summatur.

§. 22. Exemplo rem illustrabo sit $\alpha = 1$, habebitur trium sequentium serierum summa aequalis huic

$+1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ etc.
$-z - \frac{1}{4}zz - \frac{1}{9}z^3 - \frac{1}{16}z^4$ etc. $= \frac{y}{1} + \frac{yy}{4} + \frac{y^3}{9} +$ etc.
$+zlz + \frac{1}{2}z^2lz + \frac{1}{3}z^3lz$ etc.

Quia vero est $z + \frac{1}{2}zz + \frac{1}{3}z^3$ etc. $= -l(1-z) = -ly$ erit $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{y+z}{1} + \frac{y^2+z^2}{4} + \frac{y^3+z^3}{9} + \frac{y^4+z^4}{16} +$ etc. $+ lylz$, est hic $y + z = 1$, et manifestum est tales loco y vel z numeros assumi posse, vt series maxime conuergat. Id vero euenit quando $y = z$ vel vtrumque $= \frac{1}{2}$, eritque hoc casu $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc. $= 1 + \frac{1}{8} + \frac{1}{36} + \frac{1}{128} + \frac{1}{400} +$ etc. $+ \square l2$. Hoc modo summa progressionis $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc. valde prope haberi potest; est enim $l2 = \frac{1}{1.2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16}$. Summa progressionis $1 + \frac{1}{8} + \frac{1}{36}$ etc. est quam proxime $= 1,164481$ et $\square l2 = 0,480453$, ergo summa seriei $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc. est $= 1,644934$ $q.p.$ Si quis autem huius seriei summam addendis aliquot terminis initialibus determinare voluerit, plusquam mille terminos

nos addere deberet, quo noſtrum inuentum numerum reperiret.

§. 23. Ex his igitur methodum percipere licet, quomodo cuiuslibet progreſſionis, cuius termini ſunt fractiones, quarum denominatores conſtituunt progreſſionem quamcunque algebraicam, terminum ſummatorium inueniri oporteat. Equidem vt hic rem conſiderauimus, numeratores deberent eſſe quantitates conſtantes; ſed non difficulter haec methodus extendetur ad eas quoque progreſſiones, in quibus numeratores ptogreſſionem etiam quamcunque algebraicam faciunt. Propterea haec methodus ad omnes progreſſiones quarum termini generales algebraice poſſunt exponi, accommodari poteſt, eiusque ope termini ſummatorii inueniri. Excipiendi tamen ſunt caſus, quibus terminus generalis irrationalis eſt.