Hypergeometric special solutions for $d$-Painlevé equations

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Abstract

- Padé method is the simple and direct method that Painlevé equations, their Lax pairs and hypergeometric special solutions are simultaneously obtained from setting a suitable interpolation problem. In this talk, we show Padé interpolation method of additive grid applied to additive difference (d-) Painlevé equations with the affine Weyl group symmetry of type $E_7^{(1)}, E_6^{(1)}, D_4^{(1)}$ and $A_3^{(1)}$. 
Hypergeometric special solutions for \( d \)-Painlevé equations

1. Discrete Painlevé equations

2. Padé interpolation

3. Padé method

4. Case of the symmetry type \( d-E_7^{(1)} \)

5. Other cases

6. Summary and Future problems
Discrete Painlevé equations

- Continuous limit reduce to differential Painlevé equations.
- Property: Singularity confinement as a discrete analogue of Painlevé property [Gramaticos-Ramani(1991)].
- Construction:
  1. Non-autonomous version of Quispel-Roberts-Thompson mapping [Gramaticos-Ramani-Hietarinta],
  3. Deformation theory of linear difference equation [Jimbo-Sakai(1996)],

Discrete Painlevé equations are birational transformations as translation parts of the symmetry groups, etc.
Classification: elliptic(e-), multiplicative(q-) and additive(d-) difference for the affine root systems of surface/symmetry [H. Sakai(2001)].

Degeneration diagram of affine Weyl groups symmetries:

- **ell.(e-)** $E_8^{(1)}$
- **mul.(q-)** $E_8^{(1)} - E_7^{(1)} - E_6^{(1)} - D_5^{(1)} - A_4^{(1)} - (A_2 + A_1)^{(1)} - (A_1 + |\alpha|^2=14)^{(1)} - A_1^{(1)}$
- **add.(d-)** $E_8^{(1)} - E_7^{(1)} - E_6^{(1)}$ → $D_4^{(1)}(P_{VI})$ → $A_3^{(1)}(P_{V})$ → $2(A_1)^{(1)}(P_{III})$

In this talk, we read $d$-Painlevé equations with affine Weyl group symmetry of type $E_7^{(1)}$, $E_6^{(1)}$, $D_4^{(1)}$ and $A_3^{(1)}$. 
● $d$-Painlevé equations in this talk

Independent variables $(f, g) \in \mathbb{P}^1 \times \mathbb{P}^1$ and parameters $(k_1, k_2, v_1, \ldots, v_8, \delta) \in \mathbb{C}^{11}$

Equation with the symmetry type $d$-$E_7^{(1)}$ [Grammaticos-Ramani(1999)]

$$
\frac{(f + g - k_1 + k_2)(\overline{f} + g - k_1 + k_2 + \delta)}{(f + g)(\overline{f} + g)} = \frac{\Pi_{i=5}^{8}(g + k_2 - v_i)}{\Pi_{i=1}^{4}(g + v_i)},
$$

$$
\frac{(f + g - k_1 + k_2)(f + g - k_1 + k_2 - \delta)}{(f + g)(\overline{f} + g)} = \frac{\Pi_{i=5}^{8}(f - k_1 + v_i)}{\Pi_{i=1}^{4}(f - v_i)},
$$

where a constraint $2(k_1 + k_2) = \delta + \sum_{i=1}^{8} v_i$ and $\overline{\ast}$ is the time evolution $\overline{k}_1 = k_1 - \delta, \overline{k}_2 = k_2 + \delta, \overline{v}_i = v_i$.

Configuration of 8 singular points (Surface type $d$-$A_1^{(1)}$)

$P_i : (f, g) = (v_i, -v_i)_{i=1}^{4}, (k_1 - v_i, v_i - k_2)_{i=5}^{8}$. 
Equation with the symmetry type \( d-E_6^{(1)} \) [Ramani-Grammaticos-Ohta(96)]

\[
(f + g)(\bar{f} + g) = \prod_{i=1}^{4}(g + v_i) \prod_{i=5}^{6}(g + k_2 - v_i),
\]

\[
(f + g)(f + g) = \prod_{i=1}^{4}(f - v_i) \prod_{i=5}^{6}(f - k_1 + v_i).
\]

Configuration of 8 singular points

(Surface type \( d-A_2^{(1)} \))

\( P_i : (f, g) = (v_i, -v_i)_{i=1}^{4}, \)

\( (\infty, v_i - k_2)_{i=5}^{6}, \)

\( (k_1 - v_i, \infty)_{i=7}^{8}. \)
Equation with the symmetry type \( d-D_4^{(1)} \) [Ramani-Grammaticos-T.Tamizhmani-K.Tamizhmani(01)]

\[
f^\prime f = \frac{tg(g-a_4)}{(g+a_2)(g+a_1+a_2)}, \quad g+g = a_0 + a_3 + a_4 \frac{a_3}{f-1} + \frac{ta_0}{f-t}.
\]

where \( \bar{a}_0 = a_0 - 1, \bar{a}_2 = a_2 + 1, \bar{a}_3 = a_3 - 1, a_0 + a_1 + 2a_2 + a_3 + a_4 = 1. \)

Configuration of 8 singular points \((f, g)\)

(Surface type \( d-D_4^{(1)} \))

\( P_1 : (\infty, -a_2), P_2 : (\infty, -a_1 - a_2), \)

\( P_{34} : (t(1 + a_0 \epsilon), \frac{1}{\epsilon})_2, P_5 : (0, 0), \)

\( P_6 : (0, a_4), P_{78} : (1 + a_3 \epsilon, \frac{1}{\epsilon})_2. \)
Equation with the symmetry type \( d-A_3^{(1)} \) [Grammaticos-Ohta-Ramani-Sakai(98)]

\[
f \bar{f} = -\frac{g(g - a_1)}{t(g + a_2)}, \quad g + \bar{g} = a_1 + a_3 - tf + \frac{a_3}{f - 1}.
\]

where \( \bar{a}_2 = a_2 - 1, \bar{a}_3 = a_3 + 1, a_0 + a_1 + 2a_2 + a_3 = 1. \)

Configuration of 8 singular points \((f, g)\)

(Surface type \( d-D_5^{(1)} \))

\( P_1: (\infty, -a_2), P_{234}: (\frac{1}{\epsilon}, -\frac{t}{\epsilon} - a_0)_3, \)

\( P_5: (0, 0), P_6: (0, a_1), \)

\( P_{78}: (1 + a_3 \epsilon, \frac{1}{\epsilon})_2. \)
Special solutions for discrete Painlevé equations

Hypergeometric function type


Construction: Reduction to discrete Riccati equations

→ Decoupling them into two linear equations


In this talk, we construct determinant formulae of hypergeometric special solutions for the $d$-$E_7^{(1)}$, $d$-$E_6^{(1)}$, $d$-$D_4^{(1)}$ and $d$-$A_3^{(1)}$ equations by Padé interpolation.
Hypergeometric special solutions for $d$-Painlevé equations

1. Discrete Painlevé equations

2. Padé interpolation

3. Padé method

4. Case of the symmetry type $d\cdot E_7^{(1)}$

5. Other cases

6. Summary and Future problems
● **Padé interpolation**

Padé interpolation is a problem to find the polynomials $P_m(x), Q_n(x)$ of given degree $m$ and $n$ by the rational interpolation conditions

$$Y(x_s) \simeq \frac{P(x_s)}{Q(x_s)}, \quad (s = 0, 1, \ldots, m + n)$$

for given data $Y(x_s)$.

- [essentially Jacobi(1846)] a determinant expression (next slide)
- [Padé(1890)] differential version ([Frobenius],...)

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A determinant expression of $P_m(x), Q_n(x)$ [Jacobi(1846)]

For a given $Y(x_s)$ and degree($m$, $n$), the polynomials $P_m(x), Q_n(x)$ are formulated by the following determinant expressions:

$$P(x) = F(x) \det \left( \sum_{s=0}^{m+n} u_s \frac{x_s^{i+j}}{x - x_s} \right)_i j = 0,$$

$$Q(x) = \det \left( \sum_{s=0}^{m+n} u_s x_s^{i+j} (x - x_s) \right)_i j = 0.$$ 

$$u_s = \frac{Y(x_s)}{F'(x_s)}, \quad F(x) = \prod_{i=0}^{m+n} (x - x_i).$$

This expression is necessary to obtain special solutions $f$, $g$ of discrete Painlevé equations later.
Hypergeometric special solutions for $d$-Painlevé equations

1. Discrete Painlevé equations

2. Padé interpolation

3. Padé method

4. Case of the symmetry type $d-E_7^{(1)}$

5. Other cases

6. Summary and Future problems
Suitable Padé problems give Painlevé equations, Lax pairs and special solutions simultaneously through 2 kinds of 3 terms difference equations whose solutions are $\{P_m(x), Y(x)Q_n(x)\}$.

- Necessary setting

1. Choice of data $Y(x_S)$,
2. Choice of grid $x_S$ (differential, additive, $q$, $q$-quadratic, elliptic etc.),
3. Choice of time evolution direction $T$.

Suitable $Y(x)$ makes Padé method a success!
### Setting for \( d \)-Painlevé equation

Parameter: \((a_1, a_2, a_3, b_1, b_2, b_3, c, d) \in \mathbb{C}^8, (m, n) \in \mathbb{Z}_{\geq 0}^2.\)

<table>
<thead>
<tr>
<th>( d )-Painlevé eq.</th>
<th>( d-E_7^{(1)} )</th>
<th>( d-E_6^{(1)} )</th>
<th>( d-D_4^{(1)} )</th>
<th>( d-A_3^{(1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data: ( Y(x_s) )</td>
<td>( \frac{(b_1, b_2, b_3)_s}{(a_1, a_2, a_3)_s} )</td>
<td>( \frac{(b_1, b_2)_s}{(a_1, a_2)_s} )</td>
<td>( c^s \frac{(b_1)_s}{(a_1)_s} )</td>
<td>( d^s(b_1)_s )</td>
</tr>
</tbody>
</table>

a constraint \( a_1 + a_2 + a_3 + m = b_1 + b_2 + b_3 + n \) for \( d-E_7^{(1)} \).

\( (a)_i = \prod_{j=0}^{i-1} (a + j), \quad (a, b, \cdots)_i = (a)_i (b)_i \cdots. \)

(2) Grid: additive grid \( x_s = s \) \( (s = 0, 1, \ldots, m + n) \)

(3) Time evolution direction: \( T := T_{a_1} T_{b_1}, T_a : a \to a + 1, \)
\( (T := T_{a_1} T_{b_1} T_{a_3} T_{b_3} \text{ for } d-E_7^{(1)}) \quad \bar{u} := T(u), u := T^{-1}(u). \)

This setting gives \( d \)-Painlevé equations!
**Procedure of computing of Padé method**

► **Step1.** We consider the above setting, and compute 2 kinds of 3 terms relations satisfied by \( y(x) = u(x) = P_m(x), v(x) = Y(x)Q_n(x) \):

\[
\begin{vmatrix}
  y(x) & y(x + 1) & \overline{y}(x) \\
  u(x) & u(x + 1) & \overline{u}(x) \\
  v(x) & v(x + 1) & \overline{v}(x)
\end{vmatrix} = 0, \quad \begin{vmatrix}
  y(x) & \overline{y}(x) & \overline{y}(x - 1) \\
  u(x) & \overline{u}(x) & \overline{u}(x - 1) \\
  v(x) & \overline{v}(x) & \overline{v}(x - 1)
\end{vmatrix} = 0
\]

The **Casorati determinants** give contiguity relations (Lax pairs) \( L_2, L_3 \).

► **Step2.** Compatibility of \( L_2, L_3 \) ⇒ Discrete Painlevé equation

► **Step3.** \( L_2, L_3 \) and discrete Painlevé equation

⇒ 3 terms relation \( L_1 \) between \( y(x + 1), y(x), y(x - 1) \).

► **Step4.** Explicit expressions of \( P_m(x), Q_n(x) \) given by Jacobi formula

⇒ Special solutions \( f, g \)

\( Y(x_s) \) gives Lax pairs, Painlevé eq., special solutions!
Hypergeometric special solutions for \(d\)-Painlevé equations

1. Discrete Painlevé equations

2. Padé interpolation

3. Padé method

4. **Case of the symmetry type** \(d\)-\(E_7^{(1)}\)

5. Other cases

6. Summary and Future problems
• **Setting of Padé problem for** $d$-$E^{(1)}_7$ **equation**

Parameter: $(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{C}^6$, $(m, n) \in \mathbb{Z}_{\geq 0}^2$.

Conditions: $Y(x_s) \simeq P(x_s)/Q(x_s)$ \quad (s = 0, 1, \cdots, m + n),

(1) Data: $Y(x_s) = \prod_{i=1}^{3} \frac{(b_i)s}{(a_i)s}$, \quad $Y(x) = \prod_{i=1}^{3} \frac{\Gamma(a_i)\Gamma(x + b_i)}{\Gamma(x + a_i)\Gamma(b_i)}$,

with $a_1 + a_2 + a_3 + m = b_1 + b_2 + b_3 + n$,

(2) Grid: additive grid $x_s = s$,

(3) Time evolution direction: $T := T_{a_1} T_{b_1} T_{a_3} T_{b_3}$, \quad $T_a : a \to a + 1$,

$u := T(u)$, \quad $\bar{u} := T^{-1}(u)$. 
Step 1: Contiguity relations $L_2, L_3$ for $d$-$E_7$

Computing Casorati determinants

$$D_1 := \begin{vmatrix} u(x) & u(x + 1) \\ v(x) & v(x + 1) \end{vmatrix}, \quad D_2 := \begin{vmatrix} u(x) & \overline{u}(x) \\ v(x) & \overline{v}(x) \end{vmatrix}, \quad D_3 := \begin{vmatrix} u(x + 1) & \overline{u}(x) \\ v(x + 1) & \overline{v}(x) \end{vmatrix},$$

where $u(x) = P_m(x), v(x) = Y(x)Q_n(x)$, we have

$$D_1(x) = \frac{c_0 Y(x)}{\prod_{i=1}^3 (x + a_i)} \prod_{s=0}^{m+n-1} (x - s)(x - f),$$

$$D_2(x) = \frac{c_1 Y(x)}{\prod_{i=1,3} b_i(x + a_i)} \prod_{s=0}^{m+n} (x - s)(x - h),$$

$$D_3(x) = \frac{c_1 Y(x)}{b_1 b_3 \prod_{i=1}^3 (x + a_i)} \prod_{i=1,3} (x + b_i) \prod_{s=0}^{m+n-1} (x - s)(x - g),$$

where $c_0, c_1, f, g, h$ are some constants with respect to $x$. 

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Contiguity relations are rewritten by Casorati determinants $D_i$.

$L_2(x) : C_0(x - f)\overline{y}(x) - (x + a_2)(x - m - n)(x - h)y(x + 1)$

$$+ (x + b_1)(x + b_3)(x - g)y(x) = 0,$$

$L_3(x) : C_1(x - \overline{f} - 1)y(x) + (x + a_1)(x + a_3)(x - g - 1)\overline{y}(x)$

$$- x(x + b_2 - 1)(x - h)\overline{y}(x - 1) = 0,$$

where $h = g + a_2 - b_1 - b_3 - n$ and

$$C_0 = b_1 b_3 c_0 / c_1, \quad C_1 = a_1 a_3 \overline{c}_0 / c_1$$
Step 2: Painlevé equation with symmetry type $d-E_7^{(1)}$

Consider $f, g$ as unknown variables apart from Padé problem!

- Computing compatibility conditions of $L_2, L_3$, we obtain the birational equation:

$$
\frac{(f - h)(f - h + 1)}{(f - g)(f - g)} = \frac{A_2(f + 1)}{A_1(f)}, \quad \frac{(f - h)(f + 1 - h)}{(f - g)(f - g)} = \frac{A_2(h)}{A_1(g)},
$$

and

$$
A_1(x) = (x + a_2)(x + b_2)(x + 1)(x - m - n),
$$

$$
A_2(x) = \Pi_{i=1,3} (x + a_i)(x + b_i)
$$

and a constraint for the product:

$$
C_0C_1 = \frac{(a_2 - b_1 - b_3 - n)(a_2 - b_1 - b_3 - n - 1)A_1(g)}{(f - g)(f - g)}.
$$

8 singular points are on two lines $f = g, f = h$ (Surface type $d-A_1^{(1)}$).
Step 3: Linear equation $L_1$ for $d-E_7^{(1)}$ equation

Combining $L_2$, $L_3$, $d-E_7^{(1)}$ equation, $C_0C_1$ and eliminating $\overline{y}(x)$, $\overline{y}(x-1)$, $\overline{f}$, we obtain linear equation $L_1(x)$:

\[
(a_2 - b_1 - b_3 - n) \left[ \frac{(a_1 - b_2 - g + m)A_2(h)}{(a_3 + h)(h - f)(x - h)} - \frac{A_1(g)}{(f - g)(x - g - 1)} \right] y(x) \\
+ \frac{(x - h - 1)A_1(x - 1)}{(x - f - 1)(x - g - 1)} \\
\times \left[ y(x) - \frac{(x + b_1 - 1)(x + b_3 - 1)(x - g - 1)}{(x - h - 1)(x - m - n - 1)(x + a_2 - 1)} y(x - 1) \right] \\
+ \frac{(x - g)A_2(x)}{(x - f)(x - h)} \left[ y(x) - \frac{(x - h)(x - m - n)(x + a_2)}{(x + b_1)(x + b_3)(x - g)} y(x + 1) \right] = 0.
\]
\( L_1(f, g) \) equation is the curve of bidegree \((3, 2)\) passing through the following \(11(\pm 1)\) points on \( \mathbb{P}^1 \times \mathbb{P}^1 \).

8 singular points: \( f = -a_2, -b_2, -1, m + n \) on line \( f = g \),
\[
f = -a_1, -b_1, -b_3, a_1 + a_2 - b_1 - b_2 - b_3 + m - n
\]
on line \( f = g + a_2 - b_1 - b_3 - n \),

2 points: \((x, x)\) on line \( f = g \),
\[
(x-1, x-1-a_2+b_1+b_3+n) \text{ on line } f = g+a_2-b_1-b_3-n,
\]

2 points: \( y(x) = \frac{(x-h)(x-m-n)(x+a_2)}{(x+b_1)(x+b_3)(x-g)}y(x+1) \text{ at } f = x, \)
\[
y(x) = \frac{(x+b_1-1)(x+b_3-1)(x-g-1)}{(x-h-1)(x-m-n-1)(x+a_2-1)}y(x-1)
\]
at \( f = x - 1 \).
Step 4: Special solutions $f, g$ for $d\cdot E_7^{(1)}$ equation

Reconsider some constants $f, g$ of Padé problem!

Jacobi formula of additive grid($x_s = s$):

$$P(x) = \frac{F(x)}{(-(m + n))^{n+1}_{m+n}} \det \left( \sum_{s=0}^{m+n} u_s \frac{s^i+j}{x-s} \right)_{i,j=0}^n,$$

$$Q(x) = \frac{1}{(-(m + n))^{n}_{m+n}} \det \left( \sum_{s=0}^{m+n} u_s s^i+j(x-s) \right)_{i,j=0}^{n-1},$$

where

$$u_s = \frac{(-(m + n))_s}{s!} \prod_{i=1}^{3} \frac{(b_i)_s}{(a_i)_s}, \quad F(x) = \prod_{k=0}^{m+n} (x-k).$$

This expression gives a simple and direct way to obtain hypergeometric special solutions $f, g$ of discrete Painlevé equations.
Applying Jacobi formula to Casorati determinants \( D_i \), we obtain hypergeometric special solutions \( f, g \) for \( d-E_7^{(1)} \) equation:

\[
\begin{align*}
\frac{f + a_1}{f + b_1} &= \gamma \frac{T_{a_1}(\tau_{m,n}) T_{a_1}^{-1}(\tau_{m+1,n-1})}{T_{b_1}^{-1}(\tau_{m,n}) T_{b_1}(\tau_{m+1,n-1})}, \\
\frac{g + a_2}{h + a_1} &= \omega \frac{T_{a_2}(\tau_{m,n}) T_{a_2}^{-1}(\tau_{m+1,n-1})}{T_{a_1}(\tau_{m,n}) T_{a_1}^{-1}(\bar{\tau}_{m+1,n-1})},
\end{align*}
\]

where \( h = g + a_2 - b_1 - b_3 - n \) and

\[
\tau_{m,n} = \det \left( (b_1)_i (a_1 - j) \right)_{j=0}^{4F3} \left[ \frac{b_1 + i, b_2, b_3, -(m+n)}{a_1 - j, a_2, a_3} ; 1 \right]_{i,j=0}^n,
\]

\[
\gamma = -\frac{(a_1 + m + n) (a_1 - 1)^n (b_1 - 1)^n}{a_1^{n+1} b_1^n} \prod_{i=1}^{3} \frac{b_i - a_1}{a_i - b_1},
\]

\[
\omega = \frac{b_1 b_3 (b_2 - a_2)}{a_3 (b_1 - a_1) (b_3 - a_1)} \frac{(a_2 + m + n) (a_2 - 1)^n}{a_2^{n+1} (a_1 + m + n)}.
\]

and \( 4F3 \) is the generalized hypergeometric function.
Hypergeometric special solutions for $d$-Painlevé equations

1. Discrete Painlevé equations
2. Padé interpolation
3. Padé method
4. Case of $d-E_{7}^{(1)}$ equation
5. Other cases
6. Summary and Future problems
• **Case of** $d-E_6^{(1)}$ **equation**

- Setting of Padé problem

Parameter: $(a_1, a_2, b_1, b_2) \in \mathbb{C}^4$, $(m, n) \in \mathbb{Z}_0^2$.

Conditions: $Y(x_s) \simeq \frac{P(x_s)}{Q(x_s)} \quad (s = 0, 1, \cdots, m + n)$,

(1) Data: $Y(x_s) = \prod_{i=1}^{2} \frac{(b_i)_s}{(a_i)_s}, \quad Y(x) = \prod_{i=1}^{2} \frac{\Gamma(a_i)\Gamma(x + b_i)}{\Gamma(x + a_i)\Gamma(x + b_i)}$,

(2) Grid: additive grid $x_s = s$,

(3) Time evolution direction: $T := T_{a_1}T_{b_1}, \quad T_a : a \rightarrow a + 1$, 

$\overline{u} := T(u), \quad u := T^{-1}(u)$. 

Contiguity relations $L_2, L_3$ for $d-E_6^{(1)}$ equation

$L_2(x) : C_0(x - f)\bar{y}(x) - (a_2 + x)(x - m - n)y(x + 1)$

$+ (b_1 + x)(x - g)y(x) = 0,$

$L_3(x) : C_1(x - f - 1)y(x) + (x + a_1)(x - g - 1)\bar{y}(x)$

$- x(x + b_2 - 1)\bar{y}(x - 1) = 0,$

where $f$, $g$, $C_0$ and $C_1$ are some constants respect with $x$. 
Painlevé equation with the symmetry type $d$-$E_6^{(1)}$

\[
(f - g)(f - g) = \frac{(f + a_2)(f + b_2)(f + 1)(f - m - n)}{(f + a_1)(f + b_1)},
\]

\[
(f - g)(\bar{f} - g) = \frac{(g + a_2)(g + b_2)(g + 1)(g - m - n)}{(g - a_1 + b_2 - m)(g + a_2 - b_1 - n)},
\]

and a constraint for the product

\[
C_0C_1 = \frac{(g + a_2)(g + b_2)(g + 1)(g - m - n)}{(f - g)(\bar{f} - g)}.
\]

Birational equation and 8 singular points are on one curve $f = g$ and two lines $f = \infty$, $g = \infty$ (Surface type $d$-$A_2^{(1)}$):

\[
(f, g) = (-a_2, -a_2), (-b_2, -b_2), (-1, -1), (a_0 + b_0, a_0 + b_0),
\]

\[
(\infty, a_3 - b_2 + a_0), (\infty, b_1 - a_2 + b_0), (-a_1, \infty), (-b_1, \infty).
\]
Linear equation \( L_1 \) for \( d-E_6^{(1)} \) equation

\[
\left[ \frac{(g + a_2)(g + b_2)(g + 1)(g - m - n)}{(f - g)(x - g - 1)} \right. \\
\left. - (g - a_1 + b_2 - m)(g + a_2 - b_1 - n) \right] y(x) \\
+ \frac{x(x + b_2 - 1)(x - m - n - 1)(x + a_2 - 1)}{(x - f - 1)(x - g - 1)(x + b_1 - 1)(x - g - 1)} y(x - 1) \\
\times \left[ y(x) - \frac{(x + a_2 - 1)(x - m - n - 1)}{(x + a_1)(x + b_1)(x - g)} y(x - 1) \right] \\
\left. + \frac{x - f}{(x + a_2)(x - m - n)} \right] y(x + 1) = 0.
\]

\( L_1(f, g) \) equation is the curve of bidegree \((3,2)\) characterized uniquely by 11(+1) points on \( \mathbb{P}^1 \times \mathbb{P}^1 \).
Hypergeometric special solutions $f, g$ for $dE_6^{(1)}$ equation

\[
\begin{align*}
  f + a_1 &= \gamma \frac{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})}{T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})}, \\
  f + b_1 &= \omega \frac{T_{a_2}(\tau_{m,n})T_{a_2}^{-1}(\tau_{m+1,n-1})}{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})},
\end{align*}
\]

where

\[
\begin{align*}
  \tau_{m,n} &= \text{det} \left[ (b_1)_i (a_1 - j)_j \right]_3 F_2 \left[ b_1 + i, b_2, -(m + n); 1 \right]^{n}_{i,j=0}, \\
  \gamma &= -\frac{(a_1 + m + n)(a_1 - 1)^n(b_1 - 1)^n}{a_1^{n+1}b_1^n} \prod_{i=1}^{2} \frac{b_i - a_1}{a_i - b_1}, \\
  \omega &= -\frac{b_1(b_2 - a_2)(a_2 + m + n)(a_2 - 1)^n}{b_1 - a_1} \frac{a_2^{n+1}}{a_1^{n+1}},
\end{align*}
\]

and $_3F_2$ is the generalized hypergeometric function.
● Case of $d\cdot D_{4}^{(1)}$ equation

Setting of Padé problem

Parameter: $(a_1, b_1, c) \in \mathbb{C}^3$, $(m, n) \in \mathbb{Z}_{\geq 0}^2$.

Conditions: $Y(x_s) \simeq P(x_s)/Q(x_s)$ ($s = 0, 1, \ldots, m + n$),

(1) Data: $Y(x_s) = c^s \frac{(b_1)_s}{(a_1)_s}$, $Y(x) = c^x \frac{\Gamma(a_1)\Gamma(x + b_1)}{\Gamma(x + a_1)\Gamma(b_1)}$,

(2) Grid: additive grid $x_s = s$,

(3) Time evolution direction: $T := T_{a_1}T_{b_1}$, $T_a : a \rightarrow a + 1$,

$\bar{u} := T(u)$, $u := T^{-1}(u)$. 
Contiguity relations $L_2, L_3$ for $d-D_4^{(1)}$ equation

$L_2(x) : C_0(x-f)y(x) - (x-m-n)y(x+1) + \frac{1}{g}(x+b_1)y(x) = 0$

$L_3(x) : C_1(x-f-1)y(x) + \frac{1}{g}(x+a_1)y(x) - cx\bar{y}(x-1) = 0$

where $f, g, C_0$ and $C_1$ are some constants respect with $x$. 
Painlevé equation with the symmetry type \( d-D_{4}^{(1)} \)

\[ gg = (f + a_1)(f + b_1) \frac{c(f + 1)(f - m - n)}{c(f + 1)(f - m - n)}, \]

\[ f + \overline{f} = m + n - 1 - \frac{a_1 + m}{1 - cg} - \frac{b_1 + n}{1 - g}, \]

and a constraint for the product

\[ C_0C_1 = (1 - 1/g)(c - 1/g). \]

Birational equation and 8 singular points are on three lines \( f = \infty, g = 0, g = \infty \) (Surface type \( d-D_{4}^{(1)} \)):

\[ (f, g) = (-a_1, 0), (-b_1, 0), \left(1/\varepsilon, \{1 + (a_0 + a_1)\varepsilon\}/c\right)_2, \]

\[ \left(1/\varepsilon, 1 + (b_0 + b_1)\varepsilon\right)_2, (-1, \infty), (a_0 + b_0, \infty). \]
Linear equation $L_1$ for $d-D_{4(1)}$ equation

$$
\begin{align*}
&\left[c(m + n)g + (a_1 + b_1 c - cm - n) \\
&- \frac{a_1 + b_1}{g} - \frac{(x + f)(g - 1)(cg - 1)}{x + f - 1}\right]y(x) \\
&- \frac{cgx(x - m - n - 1)}{(x + a_1)(x + b_1)} \left[ y(x) - \frac{x + b_1 - 1}{y(x - 1)} \right] \\
&- \frac{g(x - m - n - 1)}{g(x - f)} \left[ y(x) - \frac{g(x - m - n)}{x + b_1} y(x + 1) \right] = 0.
\end{align*}
$$

$L_1(f, g)$ equation is the curve of bidegree $(3,2)$ characterized uniquely by 11(+1) points on $\mathbb{P}^1 \times \mathbb{P}^1$. 
Hypergeometric special solutions $f, g$ for $d-D_4^{(1)}$ equation

$$
\begin{align*}
\frac{f + a_1}{f + b_1} &= \gamma \frac{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})}{T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})}, \\
\frac{1}{g} &= 1 + \omega \frac{\tau_{m,n}\tau_{m+1,n-1}}{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})},
\end{align*}
$$

where

$$
\tau_{m,n} = \det \left[ (b_1)_{i}(a_1 - j)_{j} \right] _{2F_1} \left[ b_1 + i, -(m + n); c \right]_i j = 0^n,
$$

$$
\gamma = \frac{c(a_1 + m + n)(a_1 - 1)^n(b_1 - 1)^n}{a_1^{n+1}b_1^n}, \quad \omega = -\frac{b_1(c - 1)}{b_1 - a_1},
$$

and $2F_1$ is the Gauss hypergeometric function.
• **Case of** $d$-$A_3^{(1)}$ **equation**

- Setting of Padé problem

  Parameter: $(b_1, d) \in \mathbb{C}^2$, $(m, n) \in \mathbb{Z}^2_{\geq 0}$.

  Conditions: $Y(x_s) \approx P(x_s)/Q(x_s)$ $(s = 0, 1, \cdots, m+n)$,

  (1) Data: $Y(x_s) = d^s(b_1)_s$, $Y(x) = d^x \Gamma(x + b_1)/\Gamma(b_1)$,

  (2) Grid: additive grid $x_s = s$,

  (3) Time evolution direction: $T := T_{b_1}$, $T_a : a \to a + 1$,

    $\bar{u} := T(u)$, $u := T^{-1}(u)$. 
Contiguity relations $L_2, L_3$ for $d-A_3^{(1)}$ equation

$L_2(x) : C_0(x - f) y(x) - (x - m - n)y(x + 1) + \frac{1}{g}(b_1 + x)y(x) = 0,$

$L_3(x) : C_1(x - \bar{f} - 1)y(x) + \frac{1}{g}y(x) - dx\bar{y}(x - 1) = 0,$

where $f, g, C_0$ and $C_1$ are some constants respect with $x$. 
Painlevé equation with the symmetry type \( d-A_3^{(1)} \)

\[
gg = \frac{f + b_1}{d(f + 1)(f - m - n)}, \quad f + \bar{f} = m + n - 1 + \frac{1}{dg} - \frac{b_1 + n}{1 - g},
\]

and a constraint for the product

\[
C_0C_1 = d(1 - 1/g).
\]

Birational equation and 8 singular points are on three lines \( f = \infty, g = 0, g = \infty \) (Surface type \( d-D_5^{(1)} \)):

\[
(f, g) = (-1, \infty), (a_0 + b_0, \infty), (1/\varepsilon, \varepsilon(1 + a_0\varepsilon)/d)_3, \\
(1/\varepsilon, 1 + (b_0 + b_1)\varepsilon)_2, (-b_1, 0).
\]
Linear equation $L_1$ for $d-A_3^{(1)}$ equation

$$
\begin{align*}
&\left[ d(m + n)g + b_1d + 1 - dm - \frac{1}{g} - d(x + f)(g - 1) \right] y(x) \\
&+ \frac{dgx(x - m - n - 1)}{(x - f - 1)} \left[ y(x) - \frac{x + b_1 - 1}{g(x - m - n - 1)} y(x - 1) \right] \\
&+ \frac{x + b_1}{g(x - f)} \left[ y(x) - \frac{g(x - m - n)}{x + b_1} y(x + 1) \right] = 0.
\end{align*}
$$

$L_1(f, g)$ equation is the curve of bidegree $(3,2)$ characterized uniquely by 11(+1) points on $\mathbb{P}^1 \times \mathbb{P}^1$. 
Hypergeometric special solutions $f, g$ for $d-A_{3}^{(1)}$ equation

\[ f = -b_1 + \frac{db_1^n}{d(b_1 - 1)^n} \frac{T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})}{\tau_{m,n} \tau_{m+1,n-1}}, \]

\[ g = 1 - db_1 \frac{\tau_{m,n} \tau_{m+1,n-1}}{\tau_{m,n} \tau_{m+1,n-1}}, \]

where the determinant $\tau_{m,n}$ is given by

\[ \tau_{m,n} = \text{det} \left[ (b_1)_i (-m-n)_j \right] _{i,j=0} ^n \begin{pmatrix} 2F_0 \left( b_1 + i, - (m+n) + j, d \right) \\ 0 \end{pmatrix}, \]

and $2F_0$ is the Kummer hypergeometric function.
Hypergeometric special solutions for $d$-Painlevé equations

1. Discrete Painlevé equations

2. Padé interpolation

3. Padé method

4. Case $dE_7$

5. Other cases

6. Summary and Future problems
Summary

We applied Padé method of additive grid to $d$-Painlevé equations with the symmetry type $E_7^{(1)}$, $E_6^{(1)}$, $D_4^{(1)}$, $A_3^{(1)}$, and obtained their time evolution equations, Lax pairs and determinant formulae of hypergeometric special solutions.
Previous works of Padé method

Continuous/Discrete Painlevé equations

Degeneration diagram of affine Weyl groups symmetries:

- elliptic (e-) $E_8^{(1)}$
- multiplicative (q-) $E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + |\alpha|^2=14)^{(1)} \rightarrow A_1^{(1)} \rightarrow A_0^{(1)}$
- additive (d-) $E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_4^{(1)}_{PV1} \rightarrow A_3^{(1)}_{PV} \rightarrow 2(A_1^{(1)}_{P_{III}})^{(1)} \rightarrow A_0^{(1)}_{P_{III}}$ 

Continuous/Discrete Garnier systems

- continuous Garnier (differential grid) [Yamada(2009)],
- elliptic Garnier (elliptic grid) [Yamada(2017)].
Future problems

It may be interesting to apply Padé method to the followings.

1. $dE_8$ (additive grid).
2. multivariable $dD_4^{(1)}$ (additive grid),
3. multivariable $qE_8^{(1)}$ ($q$-quadratic grid),
4. multivariable $dE_8^{(1)}$ (additive grid),
5. We apply Padé method to higher order continuous/discrete Painlevé equations (Fuji-Suzuki, Tsuda). Previous work: Hermite-Padé [Mano-Tsuda(2014)], etc.
Thank you!