Hypergeometric special solutions for d-Painlevé equations

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Workshop on "Classical analysis, Painlevé equations"

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Abstract

▸ Padé method is the simple and direct method that Painlevé equations, their Lax pairs and hypergeometric special solutions are simultaneously obtained from setting a suitable interpolation problem. In this talk, we show Padé interpolation method of additive grid applied to additive difference (d-) Painlevé equations with the affine Weyl group symmetry of type $E_7^{(1)}$, $E_6^{(1)}$, $D_4^{(1)}$ and $A_3^{(1)}$.

1. **Discrete Painlevé equations**

2. Padé interpolation

3. Padé method

4. Case of the symmetry type d-$E_7^{(1)}$

5. Other cases

6. Summary and Future problems

• Discrete Painlevé equations

▸ Continuous limit reduce to differential Painlevé equations.

▸ Property: Singularity confinement as a discrete analogue of Painlevé property [Gramaticos-Ramani(1991)].

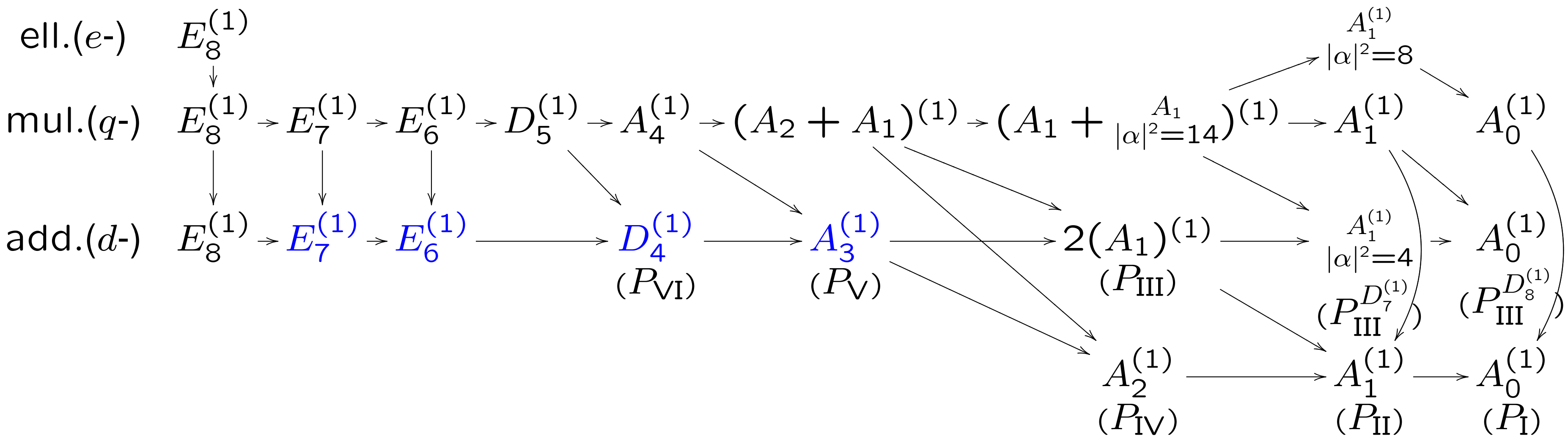
▸ Construction:

1. Non-autonomous version of Quispel-Roberts-Thompson mapping [Gramaticos-Ramani-Hietarinta],
2. Orthogonal polynomials theory [Brézin-Kazakov(1990)], [Magnus(1995)],
3. Deformation theory of linear difference equation [Jimbo-Sakai(1996)],
4. Birational representations of affine Weyl groups [Kajiwara-Noumi-Yamada].

Discrete Painlevé equations are birational transformations as translation parts of the symmetry groups, etc.

▸ Classification: elliptic(e-), multiplicative(q-) and additive(d-) difference for the affine root systems of surface/symmetry [H. Sakai(2001)].

Degeneration diagram of affine Weyl groups symmetries:



▸ In this talk, we read d-Painlevé equations with affine Weyl group symmetry of type $E_7^{(1)}$, $E_6^{(1)}$, $D_4^{(1)}$ and $A_3^{(1)}$.

• d-Painlevé equations in this talk

Independent variables $(f,g)\in\mathbb{P}^1\times\mathbb{P}^1$ and parameters $(k_1,k_2,v_1,\ldots,v_8,\delta)\in\mathbb{C}^{11}$

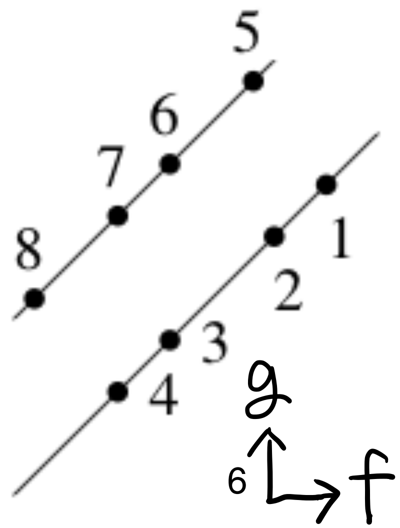
▶ Equation with the symmetry type d-$E_7^{(1)}$ [Grammaticos-Ramani(1999)]

$$\frac{(f+g-k_1+k_2)(\overline{f}+g-k_1+k_2+\delta)}{(f+g)(\overline{f}+g)}=\frac{\prod_{i=5}^{8}(g+k_2-v_i)}{\prod_{i=1}^{4}(g+v_i)},$$
$$\frac{(f+g-k_1+k_2)(f+\underline{g}-k_1+k_2-\delta)}{(f+g)(f+\underline{g})}=\frac{\prod_{i=5}^{8}(f-k_1+v_i)}{\prod_{i=1}^{4}(f-v_i)},$$

where a constraint $2(k_1+k_2)=\delta+\sum_{i=1}^{8}v_i$ and $\overline{*}$ is the time evolution $\overline{k}_1=k_1-\delta,\overline{k}_2=k_2+\delta,\overline{v}_i=v_i$.

Configuration of 8 singular points (Surface type d-$A_1^{(1)}$)

$\mathrm{P}_i:(f,g)=(v_i,-v_i)_{i=1}^{4},(k_1-v_i,v_i-k_2)_{i=5}^{8}$.



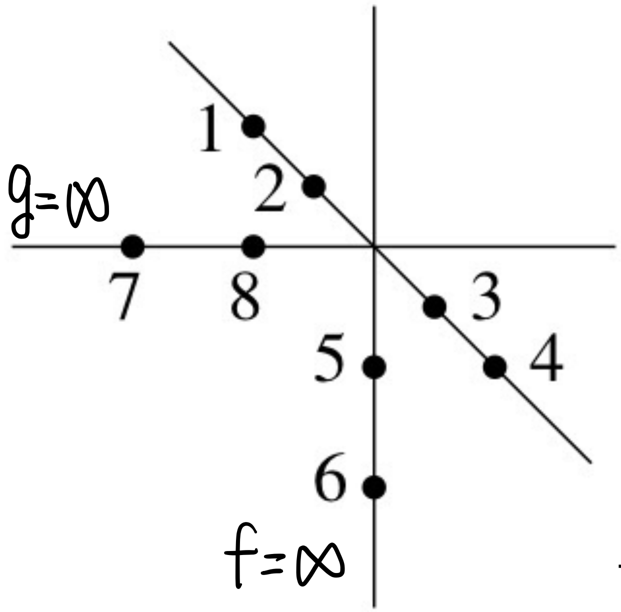
▶ Equation with the symmetry type d-$E_6^{(1)}$ [Ramani-Grammaticos-Ohta(96)]

$$(f+g)(\overline{f}+g)=\frac{\prod_{i=1}^{4}(g+v_i)}{\prod_{i=5}^{6}(g+k_2-v_i)},$$
$$(f+g)(f+\underline{g})=\frac{\prod_{i=1}^{4}(f-v_i)}{\prod_{i=5}^{6}(f-k_1+v_i)}.$$

Configuration of 8 singular points

(Surface type d-$A_2^{(1)}$)

$$\mathrm{P}_i : (f,g) = (v_i, -v_i)_{i=1}^{4},$$
$$(\infty, v_i-k_2)_{i=5}^{6},$$
$$(k_1-v_i, \infty)_{i=7}^{8}.$$



▶ Equation with the symmetry type d-$D_4^{(1)}$ [Ramani-Grammaticos-T.Tamizhmani-K.Tamizhmani(01)]

$$f\overline{f} = \frac{tg(g-a_4)}{(g+a_2)(g+a_1+a_2)}, \quad g + \underline{g} = a_0 + a_3 + a_4\frac{a_3}{f-1} + \frac{ta_0}{f-t}.$$

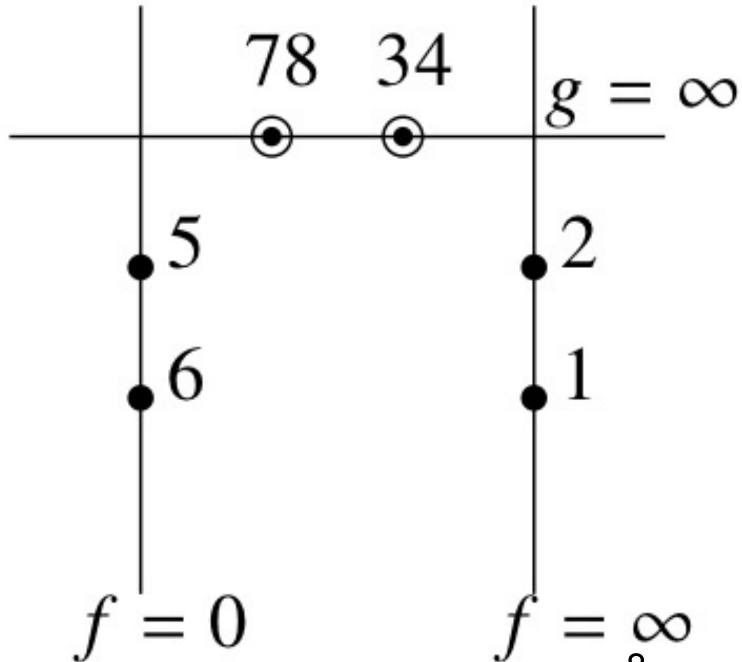
where $\overline{a}_0 = a_0 - 1$, $\overline{a}_2 = a_2 + 1$, $\overline{a}_3 = a_3 - 1$, $a_0 + a_1 + 2a_2 + a_3 + a_4 = 1$.
Configuration of 8 singular points (f, g)

(Surface type d-$D_4^{(1)}$)

$P_1 : (\infty, -a_2)$, $P_2 : (\infty, -a_1 - a_2)$,

$P_{34} : (t(1 + a_0\epsilon), \frac{1}{\epsilon})_2$, $P_5 : (0, 0)$,

$P_6 : (0, a_4)$, $P_{78} : (1 + a_3\epsilon, \frac{1}{\epsilon})_2$.



▶ Equation with the symmetry type $d\text{-}A_3^{(1)}$ [Grammaticos-Ohta-Ramani-Sakai(98)]

$$f\overline{f} = -\frac{g(g-a_1)}{t(g+a_2)}, \quad g+\overline{g} = a_1+a_3-tf+\frac{a_3}{f-1}.$$

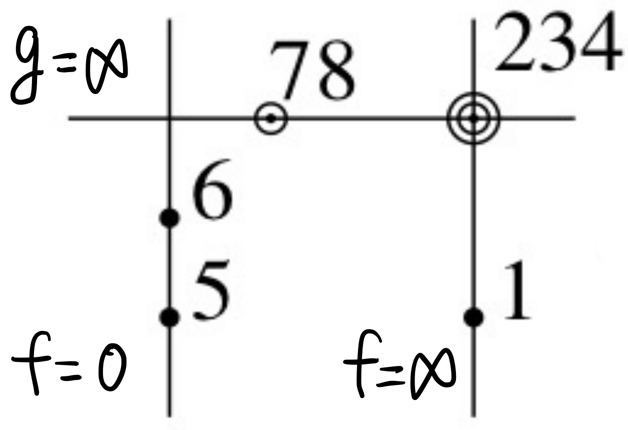
where $\overline{a}_2 = a_2 - 1$, $\overline{a}_3 = a_3 + 1$, $a_0 + a_1 + 2a_2 + a_3 = 1$.
Configuration of 8 singular points (f, g)

(Surface type $d\text{-}D_5^{(1)}$)

$P_1 : (\infty, -a_2)$, $P_{234} : (\frac{1}{\epsilon}, -\frac{t}{\epsilon} - a_0)_3$,

$P_5 : (0, 0)$, $P_6 : (0, a_1)$,

$P_{78} : (1 + a_3\epsilon, \frac{1}{\epsilon})_2$.



• Special solutions for discrete Painlevé equations

▸ Hypergeometric function type

[Kajiwara-Masuda-Noumi-Ohta-Yamada(2003, 2004)] [Kajiwara(2009)]

Construction: Reduction to discrete Riccati equations

$\rightarrow$ Decoupling them into two linear equations

▸ Rational function type [K.Tamizhmani-Grammaticos- Ramani(1993)], Algebraic function type [Grammaticos-Nijhoff-Papageorgiou-Ramani-Satsuma(1994)] [K.Tamizhmani-Ramani-Grammaticos-Ohta(1996)]

▸ In this talk, we construct determinant formulae of hypergeometric special solutions for the d-$E_7^{(1)}$, d-$E_6^{(1)}$, d-$D_4^{(1)}$ and d-$A_3^{(1)}$ equations by Padé interpolation.

Hypergeometric special solutions for d-Painlevé equations

1. Discrete Painlevé equations

2. **Padé interpolation**

3. Padé method

4. Case of the symmetry type d-$E_7^{(1)}$

5. Other cases

6. Summary and Future problems

• Padé interpolation

Padé intepolation is a problem to find the polynomials $P_m(x)$, $Q_n(x)$ of given degree m and n by the rational interpolation conditions

$$Y(x_s) \simeq \frac{P(x_s)}{Q(x_s)}, \quad (s = 0, 1, \cdots, m+n)$$

for given data $Y(x_s)$.

- [essentially Jacobi(1846)] a determinant expression(next slide)
- [Padé(1890)] differential version ([Frobenius],...)

• A determinant expression of $P_m(x)$, $Q_n(x)$ [Jacobi(1846)]

For a given $Y(x_s)$ and degree(m, n), the polynomials $P_m(x)$, $Q_n(x)$ are formulated by the following determinant expressions:

$$P(x) = F(x)\,\det\left(\sum_{s=0}^{m+n} u_s \frac{x_s^{i+j}}{x - x_s}\right)_{i,j=0}^{n},$$

$$Q(x) = \det\left(\sum_{s=0}^{m+n} u_s x_s^{i+j}(x - x_s)\right)_{i,j=0}^{n-1}.$$

$$u_s = \frac{Y(x_s)}{F'(x_s)}, \quad F(x) = \prod_{i=0}^{m+n}(x - x_i).$$

▶ This expression is necessary to obtain special solutions f, g of discrete Painlevé equations later.

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- **Padé method** [Y. Yamada(2009)]

Suitable Padé problems give Painlevé equations, Lax pairs and special solutions simultaneously through 2 kinds of 3 terms difference equations whose solutions are $\{P_m(x), Y(x)Q_n(x)\}$.

▸ Necessary setting

(1) Choice of data $Y(x_s)$,

(2) Choice of grid x_s (differential, additive, q, q-quadratic, elliptic etc.),

(3) Choice of time evolution direction T.

Suitable $Y(x)$ makes Padé method a success!

• Setting for d-Painlevé equation

Parameter: $(a_1, a_2, a_3, b_1, b_2, b_3, c, d) \in \mathbb{C}^8$, $(m, n) \in \mathbb{Z}^2_{\geq 0}$.

(1)

d-Painlevé eq.	d-$E_7^{(1)}$	d-$E_6^{(1)}$	d-$D_4^{(1)}$	d-$A_3^{(1)}$
Data : $Y(x_s)$	$\dfrac{(b_1, b_2, b_3)_s}{(a_1, a_2, a_3)_s}$	$\dfrac{(b_1, b_2)_s}{(a_1, a_2)_s}$	$c^s \dfrac{(b_1)_s}{(a_1)_s}$	$d^s (b_1)_s$

a constraint $a_1 + a_2 + a_3 + m = b_1 + b_2 + b_3 + n$ for d-$E_7^{(1)}$.

$(a)_i = \prod_{j=0}^{i-1} (a + j), \quad (a, b, \cdots)_i = (a)_i (b)_i \cdots$.

(2) Grid: additive grid $x_s = s$ $(s = 0, 1, \ldots, m + n)$

(3) Time evolution direction: $\quad T := T_{a_1} T_{b_1}$, $T_a : a \to a + 1$,

($T := T_{a_1} T_{b_1} T_{a_3} T_{b_3}$ for d-$E_7^{(1)}$) $\quad \overline{u} := T(u)$, $\underline{u} := T^{-1}(u)$.

This setting gives d-Painlevé equations !

• Procedure of computing of Padé method

▸ Step1. We consider the above setting, and compute 2 kinds of 3 terms relations satisfied by $y(x) = u(x) = P_m(x), v(x) = Y(x)Q_n(x)$:

$$L_2 : \begin{vmatrix} y(x) & y(x+1) & \overline{y}(x) \\ u(x) & u(x+1) & \overline{u}(x) \\ v(x) & v(x+1) & \overline{v}(x) \end{vmatrix} = 0, \quad L_3 : \begin{vmatrix} y(x) & \overline{y}(x) & \overline{y}(x-1) \\ u(x) & \overline{u}(x) & \overline{u}(x-1) \\ v(x) & \overline{v}(x) & \overline{v}(x-1) \end{vmatrix} = 0$$

The Casorati determinants give contiguity relations (Lax pairs)L_2, L_3.

▸ Step2. Compatibility of L_2, L_3 $\Rightarrow$ Discrete Painlevé equation

▸ Step3. L_2, L_3 and discrete Painlevé equation

$\Rightarrow$ 3 terms relation L_1 between $y(x+1), y(x), y(x-1)$.

▸ Step4. Explicit expressions of $P_m(x)$, $Q_n(x)$ given by Jacobi formula

$\Rightarrow$ Special solutions f, g

$Y(x_s)$ gives Lax pairs, Painlevé eq., special solutions !

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1. Discrete Painlevé equations

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• Setting of Padé problem for d-$E_7^{(1)}$ equation

Parameter: $(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{C}^6$, $(m, n) \in \mathbb{Z}^2_{\geq 0}$.

Conditions: $Y(x_s) \simeq P(x_s)/Q(x_s) \quad (s = 0, 1, \cdots, m+n)$,

(1) Data: $Y(x_s) = \prod_{i=1}^{3} \frac{(b_i)_s}{(a_i)_s}, \quad Y(x) = \prod_{i=1}^{3} \frac{\Gamma(a_i)\Gamma(x+b_i)}{\Gamma(x+a_i)\Gamma(b_i)}$,

with $a_1 + a_2 + a_3 + m = b_1 + b_2 + b_3 + n$,

(2) Grid: additive grid $x_s = s$,

(3) Time evolution direction: $T := T_{a_1} T_{b_1} T_{a_3} T_{b_3}, \quad T_a : a \to a + 1$,

$$\overline{u} := T(u), \quad \underline{u} := T^{-1}(u).$$

• Step 1: Contiguity relations L_2, L_3 for d-E_7

▸ Computing Casorati determinants

$$D_1 := \begin{vmatrix} u(x) & u(x+1) \\ v(x) & v(x+1) \end{vmatrix}, \quad D_2 := \begin{vmatrix} u(x) & \overline{u}(x) \\ v(x) & \overline{v}(x) \end{vmatrix}, \quad D_3 := \begin{vmatrix} u(x+1) & \overline{u}(x) \\ v(x+1) & \overline{v}(x) \end{vmatrix},$$

where $u(x) = P_m(x), v(x) = Y(x)Q_n(x)$, we have

$$D_1(x) = \frac{c_0 Y(x)}{\prod_{i=1}^{3}(x+a_i)} \prod_{s=0}^{m+n-1} (x-s)(x-f),$$

$$D_2(x) = \frac{c_1 Y(x)}{\prod_{i=1,3} b_i(x+a_i)} \prod_{s=0}^{m+n} (x-s)(x-h),$$

$$D_3(x) = \frac{c_1 Y(x)}{b_1 b_3 \prod_{i=1}^{3}(x+a_i)} \prod_{i=1,3}(x+b_i) \prod_{s=0}^{m+n-1} (x-s)(x-g),$$

where c_0, c_1, f, g, h are some constants with respect to x.

- Contiguity relations are rewritten by Casorati determinants D_i.

$$L_2(x) : C_0(x - f)\overline{y}(x) - (x + a_2)(x - m - n)(x - h)y(x + 1)$$
$$+ (x + b_1)(x + b_3)(x - g)y(x) = 0,$$

$$L_3(x) : C_1(x - \overline{f} - 1)y(x) + (x + a_1)(x + a_3)(x - g - 1)\overline{y}(x)$$
$$- x(x + b_2 - 1)(x - h)\overline{y}(x - 1) = 0,$$

where $h = g + a_2 - b_1 - b_3 - n$ and

$$C_0 = b_1 b_3 c_0/c_1, \quad C_1 = a_1 a_3 \overline{c}_0/c_1$$

• Step 2: Painlevé equation with symmetry type d-$E_7^{(1)}$

Consider f, g as unknown variables apart from Padé problem!

▸ Computing compatibility conditions of L_2, L_3, we obtain the birational equation:

$$\frac{(f-h)(f-\underline{h}+1)}{(f-g)(f-\underline{g})}=\frac{A_2(f+1)}{A_1(f)},\quad \frac{(f-h)(\overline{f}+1-h)}{(f-g)(\overline{f}-g)}=\frac{A_2(h)}{A_1(g)},$$

and

$$\begin{aligned} A_1(x)&=(x+a_2)(x+b_2)(x+1)(x-m-n),\\ A_2(x)&=\textstyle\prod_{i=1,3}(x+a_i)(x+b_i) \end{aligned}$$

and a constraint for the product:

$$C_0C_1=\frac{(a_2-b_1-b_3-n)(a_2-b_1-b_3-n-1)A_1(g)}{(f-g)(\overline{f}-g)}.$$

8 singular points are on two lines $f=g$, $f=h$ (Surface type d-$A_1^{(1)}$).

- **Step 3: Linear equation L_1 for d-$E_7^{(1)}$ equation**

▸ Combining L_2, L_3, d-$E_7^{(1)}$ equation, C_0C_1 and eliminating $\overline{y}(x), \overline{y}(x-1), \overline{f}$, we obtain linear equation $L_1(x)$:

$$(a_2 - b_1 - b_3 - n)\left[\frac{(a_1 - b_2 - g + m)A_2(h)}{(a_3 + h)(h - f)(x - h)} - \frac{A_1(g)}{(f - g)(x - g - 1)}\right] y(x)$$

$$+ \frac{(x - h - 1)A_1(x - 1)}{(x - f - 1)(x - g - 1)}$$

$$\times \left[y(x) - \frac{(x + b_1 - 1)(x + b_3 - 1)(x - g - 1)}{(x - h - 1)(x - m - n - 1)(x + a_2 - 1)} y(x - 1) \right]$$

$$+ \frac{(x - g)A_2(x)}{(x - f)(x - h)} \left[y(x) - \frac{(x - h)(x - m - n)(x + a_2)}{(x + b_1)(x + b_3)(x - g)} y(x + 1) \right] = 0.$$

▸ $L_1(f,g)$ equation is the curve of bidegree $(3,2)$ passing through the following $11(+1)$ points on $\mathbb{P}^1\times\mathbb{P}^1$.

8 singular points: $f=-a_2,-b_2,-1,m+n$ on line $f=g$,

$$f=-a_1,-b_1,-b_3,a_1+a_2-b_1-b_2-b_3+m-n$$

on line $f=g+a_2-b_1-b_3-n$,

2 points: (x,x) on line $f=g$,

$(x-1,x-1-a_2+b_1+b_3+n)$ on line $f=g+a_2-b_1-b_3-n$,

2 points: $y(x)=\dfrac{(x-h)(x-m-n)(x+a_2)}{(x+b_1)(x+b_3)(x-g)}y(x+1)$ at $f=x$,

$$y(x)=\frac{(x+b_1-1)(x+b_3-1)(x-g-1)}{(x-h-1)(x-m-n-1)(x+a_2-1)}y(x-1)$$

at $f=x-1$.

• Step 4: Special solutions f, g for d-$E_7^{(1)}$ equation

Reconsider some constants f, g of Padé problem!

▸ Jacobi formula of additive grid($x_s = s$):

$$P(x) = \frac{F(x)}{(-(m+n))_{m+n}^{n+1})} \det \left(\sum_{s=0}^{m+n} u_s \frac{s^{i+j}}{x-s} \right)_{i,j=0}^{n},$$

$$Q(x) = \frac{1}{(-(m+n))_{m+n}^{n})} \det \left(\sum_{s=0}^{m+n} u_s s^{i+j} (x-s) \right)_{i,j=0}^{n-1},$$

where

$$u_s = \frac{(-(m+n))_s}{s!} \prod_{i=1}^{3} \frac{(b_i)_s}{(a_i)_s}, \qquad F(x) = \prod_{k=0}^{m+n} (x-k).$$

This expression gives a simple and direct way to obtain hypergeometric special solutions f, g of discrete Painlevé equations.(next slide)

▶ Applying Jacobi formula to Casorati determinants D_i, we obtain hypergeometric special solutions f, g for d-$E_7^{(1)}$ equation:

$$\frac{f+a_1}{f+b_1}=\gamma\frac{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})}{T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})},$$

$$\frac{g+a_2}{h+a_1}=\omega\frac{T_{a_2}(\overline{\tau}_{m,n})T_{a_2}^{-1}(\tau_{m+1,n-1})}{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\overline{\tau}_{m+1,n-1})},$$

where $h=g+a_2-b_1-b_3-n$ and

$$\tau_{m,n}=\det\left[(b_1)_i(a_1-j)_j\,{}_4F_3\left[\begin{matrix}b_1+i,b_2,b_3,-(m+n)\\ a_1-j,a_2,a_3\end{matrix};1\right]\right]_{i,j=0}^{n},$$

$$\gamma=-\frac{(a_1+m+n)(a_1-1)^n(b_1-1)^n}{a_1^{n+1}b_1^n}\prod_{i=1}^{3}\frac{b_i-a_1}{a_i-b_1},$$

$$\omega=\frac{b_1b_3(b_2-a_2)}{a_3(b_1-a_1)(b_3-a_1)}\frac{(a_2+m+n)(a_2-1)^n}{a_2^{n+1}(a_1+m+n)}.$$

and ${}_4F_3$ is the generalized hypergeometric function.

Hypergeometric special solutions for d-Painlevé equations

1. Discrete Painlevé equations

2. Padé interpolation

3. Padé method

4. Case of d-$E_7^{(1)}$ equation

5. **Other cases**

6. Summary and Future problems

• Case of d-$E_6^{(1)}$ equation

▸ Setting of Padé problem

Parameter: $(a_1, a_2, b_1, b_2) \in \mathbb{C}^4$, $(m, n) \in \mathbb{Z}^2_{\geq 0}$.

Conditions: $Y(x_s) \simeq \dfrac{P(x_s)}{Q(x_s)} \quad (s = 0, 1, \cdots, m+n)$,

(1) Data: $Y(x_s) = \displaystyle\prod_{i=1}^{2} \frac{(b_i)_s}{(a_i)_s}, \quad Y(x) = \prod_{i=1}^{2} \frac{\Gamma(a_i)\Gamma(x+b_i)}{\Gamma(x+a_i)\Gamma(x+b_i)}$,

(2) Grid: additive grid $x_s = s$,

(3) Time evolution direction: $T := T_{a_1}T_{b_1}, \quad T_a : a \to a+1$,

$$\overline{u} := T(u), \quad \underline{u} := T^{-1}(u).$$

- Contiguity relations L_2, L_3 for d-$E_6^{(1)}$ equation

$$
\begin{aligned}
L_2(x) : C_0(x - f)\overline{y}(x) - (a_2 + x)(x - m - n)y(x + 1) \\
+ (b_1 + x)(x - g)y(x) = 0, \\
L_3(x) : C_1(x - \overline{f} - 1)y(x) + (x + a_1)(x - g - 1)\overline{y}(x) \\
- x(x + b_2 - 1)\overline{y}(x - 1) = 0,
\end{aligned}
$$

where f, g, C_0 and C_1 are some constants respect with x.

▸ Painlevé equation with the symmetry type $d\text{-}E_6^{(1)}$

$$(f-g)(f-\underline{g})=\frac{(f+a_2)(f+b_2)(f+1)(f-m-n)}{(f+a_1)(f+b_1)},$$

$$(f-g)(\overline{f}-g)=\frac{(g+a_2)(g+b_2)(g+1)(g-m-n)}{(g-a_1+b_2-m)(g+a_2-b_1-n)},$$

and a constraint for the product

$$C_0C_1=\frac{(g+a_2)(g+b_2)(g+1)(g-m-n)}{(f-g)(\overline{f}-g)}.$$

Birational equation and 8 singular points are on one curve $f=g$ and two lines $f=\infty$, $g=\infty$ (Surface type $d\text{-}A_2^{(1)}$):

$$\begin{aligned}(f,g)=&(-a_2,-a_2),(-b_2,-b_2),(-1,-1),(a_0+b_0,a_0+b_0),\\&(\infty,a_3-b_2+a_0),(\infty,b_1-a_2+b_0),(-a_1,\infty),(-b_1,\infty).\end{aligned}$$

▸ Linear equation L_1 for d-$E_6^{(1)}$ equation

$$
\begin{aligned}
&\Big[\frac{(g+a_2)(g+b_2)(g+1)(g-m-n)}{(f-g)(x-g-1)} \\
&-(g-a_1+b_2-m)(g+a_2-b_1-n)\Big]y(x) \\
&+\frac{x(x+b_2-1)(x-m-n-1)(x+a_2-1)}{(x-f-1)(x-g-1)} \\
&\times\Big[y(x)-\frac{(x+b_1-1)(x-g-1)}{(x+a_2-1)(x-m-n-1)}y(x-1)\Big] \\
&+\frac{(x+a_1)(x+b_1)(x-g)}{x-f} \\
&\times\Big[y(x)-\frac{(x+a_2)(x-m-n)}{(x+b_1)(x-g)}y(x+1)\Big]=0.
\end{aligned}
$$

$L_1(f,g)$ equation is the curve of bidegree (3,2) characterized uniquely by 11(+1) points on $\mathbb{P}^1\times\mathbb{P}^1$.

▶ Hypergeometric special solutions f, g for d-$E_6^{(1)}$ equation

$$\frac{f+a_1}{f+b_1} = \gamma \frac{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})}{T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})},$$

$$g+a_2 = \omega \frac{T_{a_2}(\overline{\tau}_{m,n})T_{a_2}^{-1}(\tau_{m+1,n-1})}{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\overline{\tau}_{m+1,n-1})},$$

where

$$\tau_{m,n} = \det\left[(b_1)_i(a_1-j)_j\,{}_3F_2\left[\begin{matrix} b_1+i, b_2, -(m+n) \\ a_1-j, a_2 \end{matrix}; 1\right]\right]_{i,j=0}^{n},$$

$$\gamma = -\frac{(a_1+m+n)(a_1-1)^n(b_1-1)^n}{a_1^{n+1}b_1^n}\prod_{i=1}^{2}\frac{b_i-a_1}{a_i-b_1},$$

$$\omega = -\frac{b_1(b_2-a_2)}{b_1-a_1}\frac{(a_2+m+n)(a_2-1)^n}{a_2^{n+1}},$$

and ${}_3F_2$ is the generalized hypergeometric function.

• Case of d-$D_4^{(1)}$ equation

▶ Setting of Padé problem

Parameter: $(a_1, b_1, c) \in \mathbb{C}^3$, $(m, n) \in \mathbb{Z}^2_{\geq 0}$.

Conditions: $Y(x_s) \simeq P(x_s)/Q(x_s) \quad (s = 0, 1, \cdots, m+n)$,

(1) Data: $Y(x_s) = c^s \dfrac{(b_1)_s}{(a_1)_s}, \quad Y(x) = c^x \dfrac{\Gamma(a_1)\Gamma(x+b_1)}{\Gamma(x+a_1)\Gamma(b_1)}$,

(2) Grid: additive grid $x_s = s$,

(3) Time evolution direction: $T := T_{a_1} T_{b_1}, \quad T_a : a \to a+1$,

$$\overline{u} := T(u), \quad \underline{u} := T^{-1}(u).$$

► Contiguity relations L_2, L_3 for d-$D_4^{(1)}$ equation

$$L_2(x) : C_0(x - f)\overline{y}(x) - (x - m - n)y(x + 1) + \frac{1}{g}(x + b_1)y(x) = 0,$$

$$L_3(x) : C_1(x - \overline{f} - 1)y(x) + \frac{1}{g}(x + a_1)\overline{y}(x) - cx\overline{y}(x - 1) = 0,$$

where f, g, C_0 and C_1 are some constants respect with x.

▸ Painlevé equation with the symmetry type $d\text{-}D_4^{(1)}$

$$g\underline{g} = \frac{(f+a_1)(f+b_1)}{c(f+1)(f-m-n)},$$

$$f+\overline{f} = m+n-1-\frac{a_1+m}{1-cg}-\frac{b_1+n}{1-g},$$

and a constraint for the product

$$C_0C_1 = (1-1/g)(c-1/g).$$

Birational equation and 8 singular points are on three lines $f=\infty$, $g=0$, $g=\infty$ (Surface type $d\text{-}D_4^{(1)}$):

$$\begin{aligned}(f,g) = {} & (-a_1,0),(-b_1,0),(1/\varepsilon,\{1+(a_0+a_1)\varepsilon\}/c)_2,\\ & (1/\varepsilon,1+(b_0+b_1)\varepsilon)_2,(-1,\infty),(a_0+b_0,\infty).\end{aligned}$$

▸ Linear equation L_1 for d-$D_4^{(1)}$ equation

$$
\begin{aligned}
&\Big[c(m+n)g + (a_1 + b_1 c - cm - n)\\
&-\frac{a_1+b_1}{g} - \frac{(x+f)(g-1)(cg-1)}{g}\Big]y(x)\\
&-\frac{cgx(x-m-n-1)}{(x-f-1)}\Big[y(x) - \frac{x+b_1-1}{g(x-m-n-1)}y(x-1)\Big]\\
&-\frac{(x+a_1)(x+b_1)}{g(x-f)}\Big[y(x) - \frac{g(x-m-n)}{x+b_1}y(x+1)\Big] = 0.
\end{aligned}
$$

$L_1(f,g)$ equation is the curve of bidegree (3,2) characterized uniquely by 11(+1) points on $\mathbb{P}^1 \times \mathbb{P}^1$.

▸ Hypergeometric special solutions f, g for d-$D_4^{(1)}$ equation

$$\frac{f+a_1}{f+b_1}=\gamma\frac{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\tau_{m+1,n-1})}{T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})},$$
$$\frac{1}{g}=1+\omega\frac{\overline{\tau}_{m,n}\tau_{m+1,n-1}}{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\overline{\tau}_{m+1,n-1})},$$

where

$$\tau_{m,n}=\det\left[(b_1)_i(a_1-j)_j\,{}_2F_1\left[\begin{matrix}b_1+i,\ -(m+n)\\ a_1-j\end{matrix};c\right]\right]_{i,j=0}^{n},$$

$$\gamma=\frac{c(a_1+m+n)(a_1-1)^n(b_1-1)^n}{a_1^{n+1}b_1^n},\quad \omega=-\frac{b_1(c-1)}{b_1-a_1},$$

and ${}_2F_1$ is the Gauss hypergeometric function.

• Case of d-$A_3^{(1)}$ equation

▸ Setting of Padé problem

Parameter: $(b_1, d) \in \mathbb{C}^2$, $(m, n) \in \mathbb{Z}^2_{\geq 0}$.

Conditions: $Y(x_s) \simeq P(x_s)/Q(x_s) \quad (s = 0, 1, \cdots, m+n)$,

(1) Data: $Y(x_s) = d^s (b_1)_s, \quad Y(x) = d^x \Gamma(x + b_1)/\Gamma(b_1)$,

(2) Grid: additive grid $x_s = s$,

(3) Time evolution direction: $T := T_{b_1}, \quad T_a : a \to a + 1$,

$$\overline{u} := T(u), \quad \underline{u} := T^{-1}(u).$$

▶ Contiguity relations L_2, L_3 for d-$A_3^{(1)}$ equation

$$L_2(x) : C_0(x - f)\overline{y}(x) - (x - m - n)y(x + 1) + \frac{1}{g}(b_1 + x)y(x) = 0,$$

$$L_3(x) : C_1(x - \overline{f} - 1)y(x) + \frac{1}{g}\overline{y}(x) - dx\overline{y}(x - 1) = 0,$$

where f, g, C_0 and C_1 are some constants respect with x.

▸ Painlevé equation with the symmetry type d-$A_3^{(1)}$

$$g\underline{g} = \frac{f+b_1}{d(f+1)(f-m-n)}, \quad f+\overline{f} = m+n-1+\frac{1}{dg}-\frac{b_1+n}{1-g},$$

and a constraint for the product

$$C_0 C_1 = d(1-1/g).$$

Birational equation and 8 singular points are on three lines $f=\infty$, $g=0$, $g=\infty$ (Surface type d-$D_5^{(1)}$):

$$(f,g) = (-1,\infty), (a_0+b_0,\infty), (1/\varepsilon, \varepsilon(1+a_0\varepsilon)/d)_3,$$
$$(1/\varepsilon, 1+(b_0+b_1)\varepsilon)_2, (-b_1, 0).$$

▶ Linear equation L_1 for d-$A_3^{(1)}$ equation

$$\left[d(m+n)g+b_1d+1-dm-\frac{1}{g}-d(x+f)(g-1)\right]y(x)$$
$$+\frac{dgx(x-m-n-1)}{(x-f-1)}\left[y(x)-\frac{x+b_1-1}{g(x-m-n-1)}y(x-1)\right]$$
$$+\frac{x+b_1}{g(x-f)}\left[y(x)-\frac{g(x-m-n)}{x+b_1}y(x+1)\right]=0.$$

$L_1(f,g)$ equation is the curve of bidegree (3,2) characterized uniquely by 11(+1) points on $\mathbb{P}^1\times\mathbb{P}^1$.

▸ Hypergeometric special solutions f, g for d-$A_3^{(1)}$ equation

$$f = -b_1 + \frac{db_1^n}{d(b_1-1)^n}\frac{T_{b_1}^{-1}(\tau_{m,n})T_{b_1}(\tau_{m+1,n-1})}{\tau_{m,n}\tau_{m+1,n-1}},$$
$$g = 1 - db_1\frac{\overline{\tau}_{m,n}\tau_{m+1,n-1}}{\tau_{m,n}\overline{\tau}_{m+1,n-1}},$$

where the determinant $\tau_{m,n}$ is given by

$$\tau_{m,n} = \det\left[(b_1)_i(-(m+n))_j\,{}_2F_0\left(\begin{matrix} b_1+i, -(m+n)+j \\ 0 \end{matrix}, d\right)\right]_{i,j=0}^n,$$

and ${}_2F_0$ is the Kummer hypergeometric function.

Hypergeometric special solutions for d-Painlevé equations

1. Discrete Painlevé equations

2. Padé interpolation

3. Padé method

4. Case d-E_7

5. Other cases

6. **Summary and Future problems**

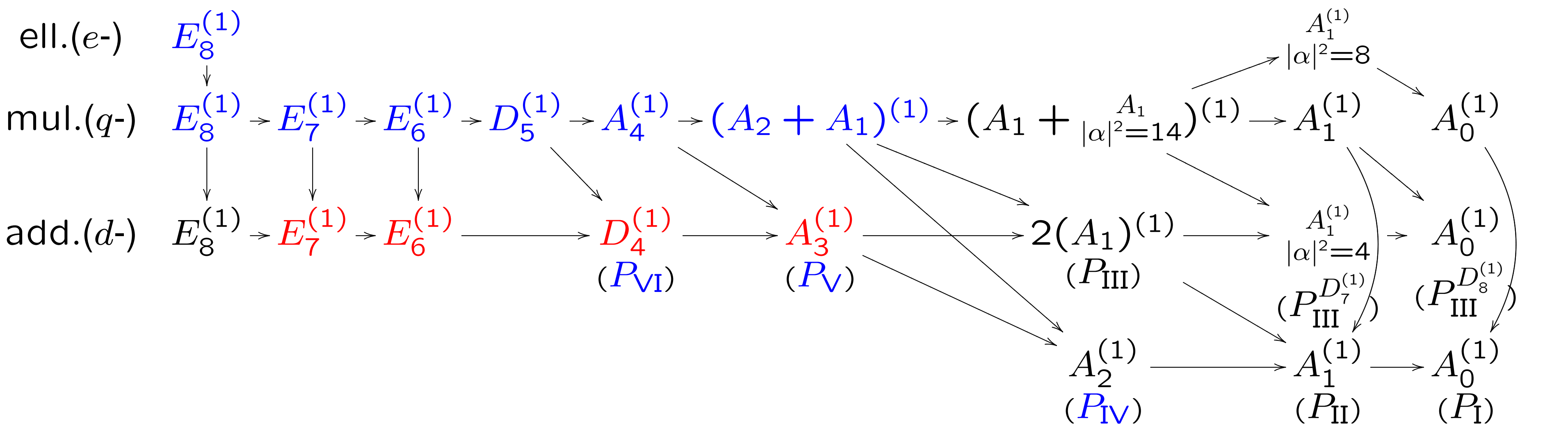
• Summary

We applied Padé method of additive grid to d-Painlevé equations with the symmetry type $E_7^{(1)}$, $E_6^{(1)}$, $D_4^{(1)}$, $A_3^{(1)}$, and obtained their time evolution equations, Lax pairs and determinant formulae of hypergeometric special solutions.

• Previous works of Padé method

► Continuous/Discrete Painlevé equations

Degeneration diagram of affine Weyl groups symmetries:



► Continuous/Discrete Garnier systems

continuous Garnier (differential grid) [Yamada(2009)],

q-Garnier (differential grid) [N-Yamada(Apr.2018, to appear)],

elliptic Garnier (elliptic grid) [Yamada(2017)].

•Future problems

▸ It may be interesting to apply Padé method to the followings.

1. d-E_8(additive grid).
2. multivariable d-$D_4^{(1)}$(additive grid),
3. multivariable q-$E_8^{(1)}$(q-quadratic grid),
4. multivariable d-$E_8^{(1)}$(additive grid),
5. We apply Padé method to higher order continuous/discrete Painlevé equations (Fuji-Suzuki, Tsuda). Previous work: Hermite-Padé [Mano-Tsuda(2014)], etc.

Thank you !